Formal Properties of Circular Traffic Queues and Cycloids.

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Report

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1 Introduction

Petri usually introduced the concept of coordination and synchronization using the regimen or organization rule for people carrying buckets to extinguish a fire [8] or by cars driving in line on a road with varying distances as shown in Figure 1 (from [9]). In the corresponding causal and infinite net, cars are represented by black tokens moving forward in time and space, whereas the gaps are moving also forward in time but in the opposite spacial direction[1].

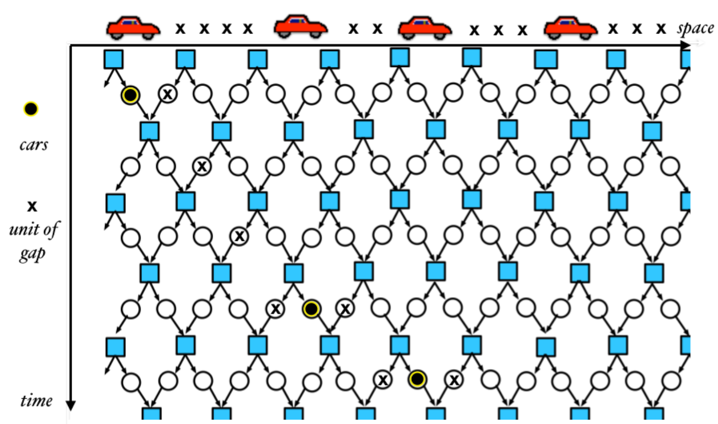

Fig. 1. Cars in Petri space.

[1] In the net, the gaps are also ordinary black tokens but are represented here by a cross to distinguish them from the cars.

As Petri always followed the principle of discrete modelling, resulting in finite structures, he defined cycloids by folding such structures with respect to space and time. While Petri introduced cycloids in [8], a more formal definition has been given in [5], [13] and [14]. As a consequence of this formalization some properties could be derived, leading for instance to a synthesis procedure based on observable parameters of the cycloid net, among which is the length of a minimal cycle. However, the concrete structure of a circular traffic queue given by a definite number of c traffic items and g gap instances was not given. As examples of circular traffic items of cars, trains, air crafts, computer tasks or electronic particles can been seen.

Motivation and results of this report are summarized as follows:

a) Presentation and formal definitions of *circular traffic queues* as elementary models of synchronisation and cooperation.
b) Definition of circular traffic queues independently from Petri net modelling by rewriting systems and transition systems.
c) Determination of safe nets, safe and secure cycloids that are behaviour equivalent to the models in b).
d) Identification of the new class of *regular cycloid systems*.
e) Introduction of the notion of *release message chain* and *cycle* as synchronization mechanism and tool for isomorphism proofs.
f) Interpretations of minimal cycloid cycles and the application of cycloid synthesis to circular traffic queue systems.
g) Relating the found cycloids to unfoldings of coloured nets.
h) Cycloid composition and iteration.

As they will be used in different contexts in this article, we recall some standard notations for set theoretical relations. If $R \subseteq A \times B$ is a relation and $U \subseteq A$ then $R[U] := \{b | \exists\, u \in U | (u, b) \in R\}$ is the *image* of U and $R[a]$ stands for $R[\{a\}]$. R^{-1} is the *inverse relation* and R^+ the *transitive closure* of R. If $R \subseteq A \times A$ is an equivalence relation then $[\![a]\!]_R$ is the *equivalence class* of the quotient A/R containing a. Furthermore $\mathbb{N}_+$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of positive integer, integer and real numbers, respectively. For $a, b \in \mathbb{Z}$ the term $a|b$ denotes, that a is a divisor of b.

2 Circular Traffic Queues

We define two classes of circular traffic queues, both with a number c of traffic items. In the first model these traffic items, going from left to right, exchange their position with a number g of different traffic items moving in the opposite direction. In the second model the opposite traffic items are anonymous and can be interpreted as gaps.

Definition 1. *A* circular traffic queue $tq(c, g)$ *is defined by two positive integers c and g. Implicitly with these integers we consider two finite and disjoint sets of traffic items $C = \{a_1, \cdots, a_c\}$ and $G = \{u_1, \cdots, u_g\}$ with cardinalities c and* *g, respectively. A* state *is a bijective index function* $ind : \{1, \cdots, n\} \to C \cup G$, *hence* $c + g = n$.

The labelled transition system $LTS(c, g) = (States, T, tr, ind_0)$ *of* $tq(c, g)$ *is defined by a set States of states, a set of transitions* $T = \{\langle\!\langle t_i, a_j \rangle\!\rangle | 1 \leq i \leq n, 1 \leq j \leq c\}$*, a* transition relation tr *and a* regular initial state ind_0*. The regular initial state is given by* $ind_0(i) = a_i$ *for* $1 \leq i \leq c$ *and* $ind_0(i) = g_{i-c}$ *for* $c < i \leq n$*. The transition relation* $tr \subseteq States \times T \times States$ *is defined by* $(ind_1, \langle\!\langle t_i, a_j \rangle\!\rangle, ind_2) \in tr \Leftrightarrow$

$$\exists\, u \in G : ind_1(i) = a_j \in C \ \wedge \ ind_1((i+1) \, mod \, n) = u \ \wedge$$
$$ind_2(i) = u \ \wedge \ ind_2((i+1) \, mod \, n) = a_j \ \wedge$$
$$ind_2(m) = ind_1(m) \ \text{ for all } \ m \notin \{i, (i+1) \, mod \, n\}$$

This is written as $ind_1 \overset{\langle\!\langle t_i, a_j \rangle\!\rangle}{\longrightarrow} ind_2$ *or* $ind_1 \to ind_2$*. A transition sequence* $ind_0 \to ind_1 \to \cdots \to ind_0$ *of minimal length, leading from the initial state* ind_0 *back to* ind_0 *is called a* recurrent *sequence. As usual* $ind_1 \overset{*}{\to} ind_2$ *denotes the reflexive and transitive closure of* tr*. We restrict the set of states to the states reachable from the initial state:* $States := \mathcal{R}(LTS(c, g), ind_0) := \{ind | ind_0 \overset{*}{\to} ind\}$*.*

A more intuitive notation would be to consider a state as a word of length n over the alphabet $C \cup G$ with distinct letters only, and the rewrite rule $au \to ua$ with $a \in C$, $u \in G$ when inside the word and $u \cdots a \to a \cdots u$ at the borders. An example of two such transitions from $tq(3, 4)$ is $u\,a\,b\,v\,w\,x\,c \to u\,a\,v\,b\,w\,x\,c \to c\,a\,v\,b\,w\,x\,u$ with $C = \{a, b, c\}$ and $G = \{u, v, w, x\}$. The model can be seen as two queues of different traffic items from C and G moving in opposite direction, but preserving their relative order in the cycle. When defining the elements of G to be indistinguishable, they can be interpreted as gaps interchanging with the traffic items from C.

Definition 2. *A* circular traffic queue with gaps $tq\text{-}g(c, g)$ *is defined as in Definition 1, with the difference that* $|G| = 1$ *and the index function ind is not bijective in general, but only on the co-image* $ind^{-1}(C)$*. In addition we require that there is at least one gap:* $ind^{-1}(G) \neq \emptyset$*. As the number g from Definition 1 is not longer needed, we use it here to define the number of gaps:* $g := n - c \geq 1$*.*

Setting $G = \{\times\}$ for $tq\text{-}g(3, 4)$ the example from above modifies to $\times ab{\times}{\times}{\times}c \to \times a{\times}b{\times}{\times}c \to c\,a{\times}b{\times}{\times}{\times}$. The model can be seen as a queue of different traffic items from C moving right if facing a gap. While the regular initial state is natural in the sense that the traffic items start without gaps in between, in a different context it is useful that the gaps are equally distributed. If for instance, as in the examples after the following definition of a *standard initial state*, the numbers c and g are even, the queue in its initial state is composed of two equal subsystems with the parameters $\frac{c}{2}$ and $\frac{g}{2}$. An analogous situation holds for larger divisors.

Definition 3. *A* standard initial state ind_0 *of a circular traffic queue* $tq(c, g)$ *is defined by the state* $ind_0(1)ind(2)_0 \cdots ind_0(n) = a_1 w_1 a_2 w_2 \cdots a_c w_c$ *with* $a_j \in C$, $w_j \in G^{r_j}$ *(set of words of length* r_j *over G) and* $r_j = |\{\, x \in \mathbb{N} \mid j - 1 < \frac{c}{g} \cdot x \leq j\}|$ *for* $1 \leq j \leq c$ *.*

To be a consistent definition it should be verified that all the r_i sum up to g.

Lemma 4. *If ind is the standard initial state of Definition 3 then* $\sum_{i=j}^{c} r_j = g$.

Proof. This follows from the observation that the intervals in the definition of r_i are disjoint and define all together a set $\{ x \in \mathbb{N} \mid 0 < \frac{c}{g} \cdot x \leq c\} = \{ x \in \mathbb{N} \mid 0 < x \leq g\}$ of cardinality g. □

To give an example, we consider the case $c = 4, g = 6$. We obtain $(r_1, r_2, r_3, r_4) = (1, 2, 1, 2)$ and $a_1 \times a_2 \times\times a_3 \times a_4 \times\times$ for the resulting standard initial state. The sets in the definitions of r_1 and r_2 are $\{1\}$ and $\{2, 3\}$, respectively. For $c = 6, g = 4$ we obtain $a_1 a_2 \times a_3 \times a_4 a_5 \times a_6 \times$.

Theorem 5. *Let be $\Delta = gcd(c, g)$ the greatest common divisor of c and g.*
a) The length of each recurrent sequence (Definition 1) of a circular traffic queue with gaps tq-g(c, g) is $\Gamma(c, g) := c \cdot (c + g)$.
b) The length of each recurrent sequence of a circular traffic queue $tq(c, g)$ is $\Xi(c, g) := \frac{g}{\Delta} \cdot (c + g) \cdot c$.

Proof. In both cases we start with a regular initial state
$ind_0 = a_1 a_2 \cdots a_c u_1 u_2 \cdots u_g$ with $a_j \in C$ and $u_h \in G$.

a) In the first case all u_i equal $\times$. To reach the initial state ind_0 for the first time, each traffic item $a_i \in C$ has to make $n = c + g$ steps. Hence in total, we have $\Gamma(c, g) := c(c + g)$. Each transition of a traffic item $a_j, 1 \leq j \leq c$ is occurring in a position $i, 1 \leq i \leq n$. It could be unambiguously labeled by $[t_i, a_j]$. Hence the number of transitions is $\Gamma(c, g)$.

b) Since the model $tq(c, g)$ is symmetric with respect to G and C, as well as the result $\Xi(c, g) := \frac{1}{\Delta} c(c+g)g$ to be proved, it is sufficient to consider the case $g \geq c$. Furthermore we assume $g > 1$ since for $g = 1$ we also have $c = 1$ and the problem reduces to case a) of the proof.

The recurrent sequence to be constructed is split into several pieces. We start by shifting all cars to the end of the queue requiring g steps for each and $g \cdot c$ steps in total:

$$ind_0 = a_1 a_2 \cdots a_c u_1 u_2 \cdots u_g \quad \xrightarrow{g \cdot c - times} u_1 u_2 \cdots u_g a_1 a_2 \cdots a_c \tag{1}$$

Then we move the cars back to their initial position, needing $c \cdot c$ steps:

$$u_1 u_2 \cdots u_g a_1 a_2 \cdots a_c \xrightarrow{c \cdot c} a_1 a_2 \cdots a_c u_{c+1} \cdots u_g u_1 \cdots u_c \tag{2}$$

Ignoring the individual names of the elements $u_h \in G$ we have the situation described in case a) of the proof needing $c \cdot g + c \cdot c = c \cdot (c + g)$ steps. But here in case b) of the proof the initial state is not yet reached since the items u_h are not necessary in their initial order. Therefore the $c \cdot (c + g)$ steps have to be repeated in a number of different levels. In the following, the step from level k to $k + 1$ is shown. The letter $u_{i^k_j}$ denotes that the item u_{i_j} of level k is in position $ind^{-1}(u_{i_j}) = j$.

$$a_1 a_2 \cdots a_c u_{i_1^k} u_{i_2^k} \cdots u_{i_g^k} \xrightarrow{c(c+g)} a_1 a_2 \cdots a_c u_{i_1^{k+1}} u_{i_2^{k+1}} \cdots u_{i_g^{k+1}} \quad \text{with } u_{i_1^{k+1}} = u_{(i_1^k + c)\, mod\, g} \tag{3}$$

We now consider the sequence $u_{i_1^1} u_{i_1^2} \cdots u_{i_1^k} \cdots$ of the first items from G for all levels k in equation (3): $u_{i_1^1} u_{i_1^2} \cdots u_{i_1^k} \cdots$ starting with $u_{i_1^1} = u_1$ from ind_0. The initial state is reached when $u_{i_1^k} = u_{i_1^1} = u_1$ for the first time. By induction, beginning with equations (1) and (2): $u_{i_1^1} = u_1, u_{i_1^2} = u_{1+c}$ and by the induction step from equation (3): $u_{i_1^{k+1}} = u_{(i_1^k + c)\, mod\, g}$ we conclude $u_{i_1^k} = u_{(k \cdot c+1)\ mod\ g}$. Hence to determine the value of k for a recurrent sequence we have to find the smallest nontrivial solution in the following equation:

$$(k \cdot c + 1)\, mod\, g = 1 \tag{4}$$

Next we prove $k = \frac{g}{x}$ to be a solution of (4), where $x \in \mathbb{N}$. This is proved by: $(\frac{g}{x} \cdot c + 1)\ mod\ g = ([(\frac{c}{x} \cdot g)\ mod\ g] + [1\ mod\ g])\ mod\ g = (0+1)\ mod\ g = 1$ since $g > 1$. In this calculation $\frac{c}{x}$ has to be an integer. Therefore x is a divisor of c. The same holds for $k = \frac{g}{x}$ and x has also to be a divisor of g. For a minimal non-trivial solution we obtain $x = gcd(c, g) = \varDelta$ and $k = \frac{g}{\varDelta}$. Recall that k is the number times the step from equation (3) has to be repeated to obtain a recurrent transition sequence. This gives the result $\varXi(c, g) = k \cdot c \cdot (c+g) = \frac{c \cdot (c+g) \cdot g}{\varDelta}$. □

The proof has shown that a sequence has to be repeated $\frac{g}{\varDelta}$ times after having reached a state where all traffic items from C are in their initial order. To obtain an adequate labelling of the transitions we add a counter $k, 0 \le k < r$ to represent the repetitive behaviour. The counter is implemented as an exponent of the names of traffic items a_j^k and transitions t_v^k. Each time the traffic item a_j starts a new round in its (initial) position $j \in \{1, \cdots, c\}$ (with respect to the regular initial state) the counter is increased. With the values of i and k in the following definition, for each traffic item a_j a number $p = n \cdot r$ of process states is reached. Later we will restrict to the cases $r = 1$ (no repetition) and $r = \frac{g}{\varDelta}$.

Definition 6. *Let be $r \in \mathbb{N}_+$ and $C^r := \{a_j^k | a_j \in C, 1 \le k < r\}$. A (r-repetitive)* labelled transition system $LTS_p(c, g) = (States, T, tr, ind_0)$ *with $p = r \cdot n$ and $c, g, n = c + g \in \mathbb{N}_+$ is defined by a set States of states as in Definition 1 with C replaced by C^r, a set of transitions $T = \{\langle\langle t_v^k, a_j \rangle\rangle | 1 \le v \le p, 1 \le k < r, 1 \le j \le c\}$. The transition relation $tr \subseteq States \times T \times States$ is defined by* $(ind_1, \langle\langle t_v^k, a_j\rangle\rangle, ind_2) \in tr \Leftrightarrow$

$$\begin{aligned}
&\exists\, u \in G\ \ \exists\, i \in \{1, \cdots, n\} : v = k \cdot n + i \wedge \\
&ind_1(i) = a_j^k \in C^r\ \wedge\ ind_1((i+1)\, mod\, n) = u\ \wedge \\
&ind_2(i) = u\ \wedge\ [ind_2((i+1)\, mod\, n) = a_j^k\ \textbf{\textit{if}}\ i \neq j\ \textbf{\textit{else}}\ a_j^{(k+1)\, mod\, r}]\ \wedge \\
&ind_2(m) = ind_1(m)\ \text{ for all }\ m \notin \{i, (i+1)\, mod\, n\}
\end{aligned}$$

In particular, we consider the special cases tq-2$(c, g) := LTS_p(c, g)$ with $p = \frac{g}{\varDelta} \cdot n$ and tq-1$(c, g) := LTS_n(c, g)$. In the latter cases we have $r = 1$ and the labelling of the transitions can be simplified to $\langle\langle t_i, a_j \rangle\rangle$.

For preparing the modelling of alternative formalisms, in particular of Petri nets in Section 4, we give a specification of circular traffic queues by their properties. To be free from a more sequential specification we prefer a message-oriented formulation. This is similar to Petri's notion of a *permit signal* in [10].

Definition 7. *A circular traffic queue tq-*$1(c,g)$ *respectively tq-*$2(c,g)$ *has the following properties. Next,* $u \in G$ *denotes a gap in the case of tq-*$1(c,g)$ *and a traffic item in the case of tq-*$2(c,g)$*.*
a) Each traffic item $a \in C$ *and* $u \in G$ *is in exactly in one of* $n = c + g$ *positions.*
b) Each traffic item $a \in C$ *can make a step from position* $i \in \{1, \cdots, n\}$ *to position* $(i+1) \bmod n$*, if it has received a release message from* $u \in G$ *in position* $(i+1) \bmod n$*. After this step the* $u \in G$ *is in position* i*.*
c)The length of recurrent transition sequences is $\Gamma(c,g) = c \cdot n$ *and* $\Xi(c,g) := \frac{g}{\Delta} \cdot c \cdot n$*, respectively.*
d) tq-$2(c,g)$ *consists of* $\frac{g}{\Delta}$ *copies of transition systems of type tq-*$1(c,g)$*. After* $c \cdot n$ *steps each transition sequence enters the next of these copies.*

This specification is denoted as a definition, but requires some justification. The items a), b) and c) follow from the definitions of a circular traffic queue. The supplement concerning *tq*-$2(c,g)$ follows from the proof of Theorem 5.

3 Cycloids

In this section (Petri-) nets and cycloids are defined. Also results which are used in this article are cited from [13] and [14]. New with respect to these articles are results on regular cycloids and the use of matrix algebra in proofs.

Definition 8. *A* net $\mathcal{N}$ *is defined by a triple* (S,T,F) *where* S *is an non-empty set, called set of* state elements *or* places*, a non-empty set* T *of* transitions *and a* flow relation F*, with the following restrictions:* $S \cap T = \emptyset$ *and* $F \subseteq S \times T \cup T \times S$*.*
An element from $X := S \cup T$ *is said to be a* net element *of* $\mathcal{N}$*. Given two nets* $\mathcal{N} = (S,T,F)$ *and* $\mathcal{N}' = (S',T',F')$*, a net morphism [11] is a mapping* $f : X \to X'$*, denoted* $f : \mathcal{N} \to \mathcal{N}'$*, if* $f(F \cap (S \times T)) \subseteq (F' \cap (S' \times T')) \cup id$ *and* $f(F \cap (T \times S)) \subseteq (F' \cap (T' \times S')) \cup id$*. It is an isomorphism if it is bijective and the inverse mapping* f^{-1} *is also a net morphism. In this case* $\mathcal{N}$ *and* $\mathcal{N}'$ *are said to be isomorphic, which is denoted by* $\mathcal{N} \simeq \mathcal{N}'$*.*
${}^\bullet x := F^{-1}[x]$, $x^\bullet := F[x]$ *denote the input and output elements of an element* x*, respectively.*

A transition $t \in T$ *is* active *in a marking* $M \subseteq S$ *if* ${}^\bullet t \subseteq M$ *and* $t^\bullet \cap M = \emptyset$*.*

If a transition $t \in T$ *is* active *in a marking* $M \subseteq S$*, the* follower marking M' *is defined and given by* $M' = M \setminus {}^\bullet t \cup t^\bullet$*. In this case we write* $M \xrightarrow{t} M'$*. As usual, the reflexive and transitive closure of this relation is denoted* $M \xrightarrow{*} M'$ *(*M' *is reachable from* M*.) A net together with an initial marking* $M_0 \subseteq S$ *is called a* net-system $(\mathcal{N}, M_0)$ *with its* reachability set $\mathcal{R}(\mathcal{N}, M_0) := \{M | M_0 \xrightarrow{*} M\}$*. The* reachabilty graph $\mathcal{RG}(\mathcal{N}, M_0) = (\mathcal{R}(\mathcal{N}, M_0), \rightarrow)$ *is defined by the reachability set as the set of nodes and the follower marking relation* $\rightarrow$ *as its set of arrows.*

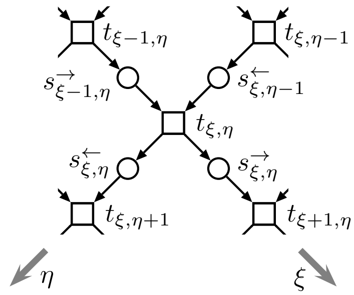

Fig. 2. Denomination of Petri space elements

Definition 9. *A Petri space is defined by the net*

$\mathcal{PS}_1 := (S_1, T_1, F_1)$ *where*

$S_1 = S_1^{\rightarrow} \cup S_1^{\leftarrow}$, $S_1^{\rightarrow} = \{s^{\rightarrow}_{\xi,\eta} \mid \xi,\eta \in \mathbb{Z}\}$, $S_1^{\leftarrow} = \{s^{\leftarrow}_{\xi,\eta} \mid \xi,\eta \in \mathbb{Z}\}$, $S_1^{\rightarrow} \cap S_1^{\leftarrow} = \emptyset$,

$T_1 = \{t_{\xi,\eta} \mid \xi,\eta \in \mathbb{Z}\}$,

$F_1 = \{(t_{\xi,\eta}, s^{\rightarrow}_{\xi,\eta}) \mid \xi,\eta \in \mathbb{Z}\} \cup \{(s^{\rightarrow}_{\xi,\eta}, t_{\xi+1,\eta}) \mid \xi,\eta \in \mathbb{Z}\} \cup$

$\{(t_{\xi,\eta}, s^{\leftarrow}_{\xi,\eta}) \mid \xi,\eta \in \mathbb{Z}\} \cup \{(s^{\leftarrow}_{\xi,\eta}, t_{\xi,\eta+1}) \mid \xi,\eta \in \mathbb{Z}\}$.

$X_1 := S_1 \cup T_1$. $S_1^{\rightarrow}$ *is called set of* forward places *and* $S_1^{\leftarrow}$ *the set of* backward places. $\overset{\rightarrow}{{}^{\bullet}}t_{\xi,\eta} = s^{\rightarrow}_{\xi-1,\eta}$ *is the forward input place of* $t_{\xi,\eta}$ *and in the same way* $\overset{\leftarrow}{{}^{\bullet}}t_{\xi,\eta} := s^{\leftarrow}_{\xi,\eta-1}$, $t^{\overset{\rightarrow}{\bullet}}_{\xi,\eta} := s^{\rightarrow}_{\xi,\eta}$ *and* $t^{\overset{\leftarrow}{\bullet}}_{\xi,\eta} := s^{\leftarrow}_{\xi,\eta}$.

Definition 10. *A* cycloid *is a net* $\mathcal{C}(\alpha,\beta,\gamma,\delta) = (S,T,F)$*, defined by parameters* $\alpha,\beta,\gamma,\delta \in \mathbb{N}_+$*, by a quotient of the Petri space* $\mathcal{PS}_1 := (S_1, T_1, F_1)$ *(Definition 9), with respect to the equivalence relation* $\equiv\, \subseteq X_1 \times X_1$ *with*

$\equiv[S_1^{\rightarrow}] \subseteq S_1^{\rightarrow}$, $\equiv[S_1^{\leftarrow}] \subseteq S_1^{\leftarrow}$, $\equiv[T_1] \subseteq T_1$,

$x_{\xi,\eta} \equiv x_{\xi+m\alpha+n\gamma,\, \eta-m\beta+n\delta}$ *for all* $\xi,\eta,m,n \in \mathbb{Z}$, $X = X_1/_{\equiv}$

$[\![x]\!]_{\equiv}\, F\, [\![y]\!]_{\equiv} \Leftrightarrow \exists x' \in [\![x]\!]_{\equiv}\ \exists y' \in [\![y]\!]_{\equiv} : x'\, F_1\, y'$ *for all* $x,y \in X_1$.

The matrix $\mathbf{A} = \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix}$ *is called the matrix of the cycloid. The value of its determinant is* $A = det(\mathbf{A}) = \alpha\delta + \beta\gamma = |T|$*. As it equals the number of transitions it is called the area of* $\mathcal{C}(\alpha,\beta,\gamma,\delta)$.

Cycloids are T-nets[2]*, i.e. nets with* $|{}^{\bullet}s| = |s^{\bullet}| = 1$ *for all places* $s \in S$.

Isomorphic nets are called as cycloids, as well. The embedding of a cycloid in the Petri space is called fundamental parallelogram *(see Figure 3, ignore the tokens for the moment). If the cycloid is represented as a net* $\mathcal{N}$ *(without explicitly giving the parameters* $\alpha,\beta,\gamma,\delta$*) we call it a cycloid in net form* $\mathcal{C}(\mathcal{N})$.

For proving the equivalence of two points in the Petri space the following procedure is useful.

[2] Also called marked graphs.

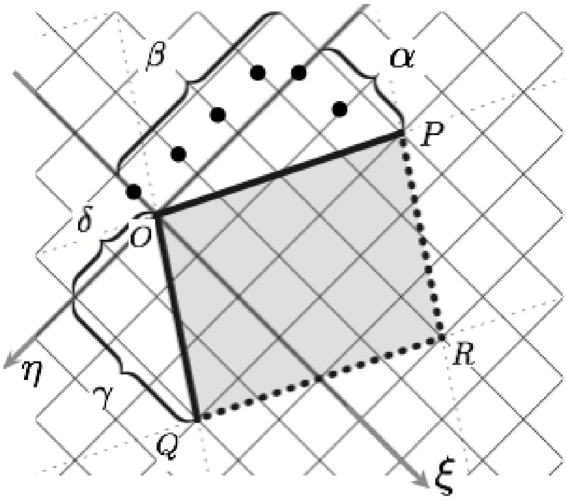

Fig. 3. Fundamental parallelogram of $\mathcal{C}(2,4,3,2)$

Definition 11. *For two points* $\mathbf{x}_1, \mathbf{x}_2 \in X_1$ *and the vector* $\mathbf{v} = \mathbf{x_2} - \mathbf{x_1}$ *we define a parameter vector* $\pi(\mathbf{v}) = (m, n)$ *of the parameters* m *and* n *of Definition 10.*

Theorem 12. *The parameter vector has the value* $\pi(\mathbf{v}) = \frac{1}{A} \cdot \mathbf{B} \cdot \mathbf{v}$ *where* A *is the area and* $\frac{1}{A}\mathbf{B} = \frac{1}{A}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}$ *is the inverse of the cycloid matrix* $\mathbf{A}$*. To decide the equivalence* $\mathbf{x}_1 \equiv \mathbf{x}_2$ *it is suffient to check whether* $\pi(\mathbf{v}) = \pi(\mathbf{x_2} - \mathbf{x_1}) \in \mathbb{Z} \times \mathbb{Z}$.

Proof. For $\mathbf{x}_1 := \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \mathbf{x}_2 := \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}, \mathbf{v} := \mathbf{x}_2 - \mathbf{x}_1$ from Definition 10 we obtain in matrix form: $\mathbf{x}_1 \equiv \mathbf{x}_2 \Leftrightarrow \exists\, m, n \in \mathbb{Z} : \begin{pmatrix} \xi_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 + m\alpha + n\gamma \\ \eta_2 - m\beta + n\delta \end{pmatrix} \Leftrightarrow$
$\exists\, m, n \in \mathbb{Z} : \mathbf{v} = \begin{pmatrix} \xi_2 - \xi_1 \\ \eta_2 - \eta_1 \end{pmatrix} = \begin{pmatrix} m\alpha + n\gamma \\ -m\beta + n\delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ -\beta & \delta \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \mathbf{A} \begin{pmatrix} m \\ n \end{pmatrix} \Leftrightarrow$
$\begin{pmatrix} m \\ n \end{pmatrix} = \mathbf{A}^{-1}\mathbf{v} \in \mathbb{Z} \times \mathbb{Z}$.

It is well-known that $\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})}\mathbf{B}$ with $\mathbf{B} = \begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}$ if $det(\mathbf{A}) > 0$ (see any book on (linear) algebra, [6] for example). The condition $det(\mathbf{A}) = A = \alpha\delta + \beta\gamma > 0$ is satisfied by the definition of a cycloid. □

Using Theorem 12 we derive the parameters m and n for the corners of the fundamental parallelogram.

Theorem 13. *With respect to the origin the parameters for the corners* O*,* P*,* Q *and* R *are* $(0,0)$*,* $(1,0)$*,* $(0,1)$ *and* $(1,1)$*, respectively.*

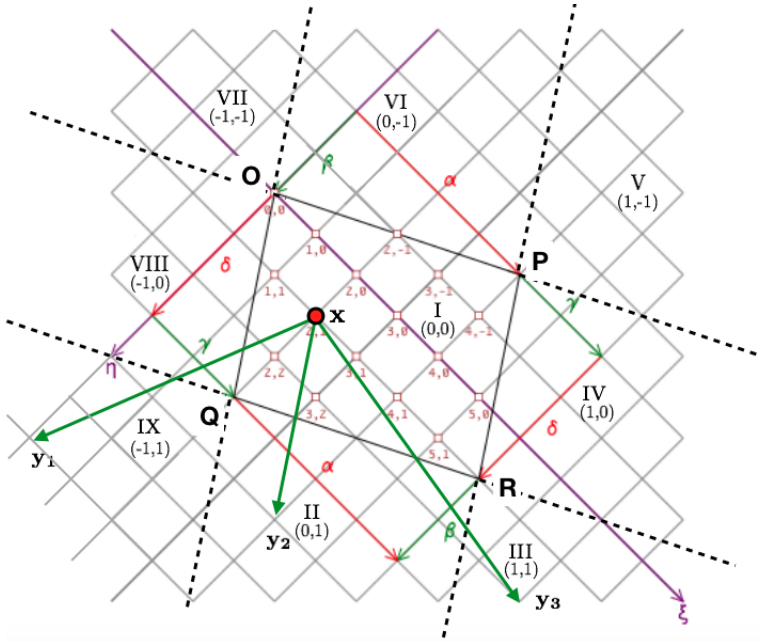

Fig. 4. Fundamental parallelograms with parameters for equivalent elements.

Proof. We give the proof in the case of R, while the other cases are proved in a similar way. Using the coordinates $\begin{pmatrix}\alpha+\gamma\\ \delta-\beta\end{pmatrix}$ of R we obtain by Theorem 12: $\pi(\begin{pmatrix}\alpha+\gamma\\ \delta-\beta\end{pmatrix}) = \frac{1}{A}\begin{pmatrix}\delta & -\gamma\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}\alpha+\gamma\\ \delta-\beta\end{pmatrix} = \frac{1}{A}\begin{pmatrix}\delta(\alpha+\gamma)-\gamma(\delta-\beta)\\ \beta(\alpha+\gamma)+\alpha(\delta-\beta)\end{pmatrix} = \frac{1}{A}\begin{pmatrix}A\\ A\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$ □

To give an example, in Figure 4 the fundamental parallelogram (Roman number I) of the cycloid $\mathcal{C}(4,2,2,3)$ with corners O, P, R, and Q is given together with parts of the eight neighbouring fundamental parallelograms (Roman numbers II - IX). Starting from a transition $\mathbf{x}$ of the fundamental parallelogram the equivalent transition in such a neighbouring fundamental parallelogram is obtained by the corresponding parameters (m, n) placed below the romain number. For instance, the transitions $\mathbf{y_1}$, $\mathbf{y_2}$ and $\mathbf{y_3}$ are obtained by:

$$\text{IX)}\ \mathbf{y_1} = \mathbf{x} + \mathbf{A}\begin{pmatrix}m\\ n\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix} + \begin{pmatrix}4 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 6\end{pmatrix}$$

$$\text{II)}\ \mathbf{y_2} = \mathbf{x} + \mathbf{A}\begin{pmatrix}m\\ n\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix} + \begin{pmatrix}4 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}4\\ 4\end{pmatrix}$$

$$\text{III)}\ \mathbf{y_3} = \mathbf{x} + \mathbf{A}\begin{pmatrix}m\\ n\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix} + \begin{pmatrix}4 & 2\\ -2 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}8\\ 2\end{pmatrix}$$

Note that the added vectors are the *repetitive vectors* between the corners of the fundamental parallelogram: $\mathbf{y_1} - \mathbf{x} = \overrightarrow{PQ}, \mathbf{y_2} - \mathbf{x} = \overrightarrow{OQ}, \mathbf{y_3} - \mathbf{x} = \overrightarrow{OR}$.

Theorem 14 ([13][14]). *The following cycloids are isomorphic to $\mathcal{C}(\alpha, \beta, \gamma, \delta)$:*

a) $\mathcal{C}(\beta, \alpha, \delta, \gamma)$, (This cycloid will be called the dual cycloid *of $\mathcal{C}(\alpha, \beta, \gamma, \delta)$.)*
b) $\mathcal{C}(\alpha, \beta, \gamma - x \cdot \alpha, \delta + x \cdot \beta)$ if $x \in \mathbb{N}_+$ and $\gamma > x \cdot \alpha$,
c) $\mathcal{C}(\alpha, \beta, \gamma + x \cdot \alpha, \delta - x \cdot \beta)$ if $x \in \mathbb{N}_+$ and $\delta > x \cdot \beta$.

Proof. Part a) has been proved in [13][14] as well b) and c) for the special case of $x = 1$. The current form is derived by iterating the result. □

Definition 15. *For a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ constructed as a quotient from the Petri space $\mathcal{PS}_1 = (S_1, T_1, F_1)$ by the equivalence relation $\equiv$ we define a cycloid-system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ or $\mathcal{C}(\mathcal{N}, M_0)$ by adding the standard initial marking*

$$\begin{aligned} M_0 = &\{s^{\rightarrow}_{\xi,\eta} \in S^{\rightarrow}_1 \mid \beta\xi + \alpha\eta \leq 0 \ \wedge \ \beta(\xi+1) + \alpha\eta > 0\} /_{\equiv} \ \cup \\ &\{s^{\leftarrow}_{\xi,\eta} \in S^{\leftarrow}_1 \mid \beta\xi + \alpha\eta \leq 0 \ \wedge \ \beta\xi + \alpha(\eta+1) > 0\} /_{\equiv} \end{aligned}$$

As in the case of circular traffic queues we define a regular initial marking for cycloids. It is characterized by the absence of gaps between the traffic items. In the case of the standard initial marking for every cycloid the transition $t_{1,0}$ is active (Lemma 4.2 of [13],[14]). For a regular initial marking it is semi-active only, as there is no gap in the following position. The corresponding token is in $s^{\rightarrow}_{-1,0}$. The next traffic item is in position 2 of the queue. The corresponding token is in $s^{\rightarrow}_{-1,-1}$ (see Figure 3). This argument holds for all but the last transition. The corresponding traffic item is in position β of the queue. The corresponding token is in $s^{\rightarrow}_{-1,-\beta+1}$. The last transition is facing all α gaps and can therefore make α steps without another traffic item moving. Therefore we have tokens corresponding to the gaps in the places $s^{\leftarrow}_{0,-\beta}, \cdots, s^{\leftarrow}_{\alpha-1,-\beta}$ (see Figure 3). Equivalent places within the fundamental parallelogram can be computed. For instance, the regular initial marking of $\mathcal{C}(4, 3, 3, 3)$ is represented within the fundamental parallelogram by highlighted places in Figure 8.

Definition 16. *For a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ a regular initial marking is defined by a number of β forward places $\{s^{\rightarrow}_{-1,i} \mid 0 \geq i \geq 1-\beta\}$ and a number of α backward places $\{s^{\leftarrow}_{i,-\beta} \mid 0 \leq i \leq \alpha - 1\}$.*

Theorem 17 ([13][14]). *The length of a minimal cycle of a cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ is*

$$cyc(\alpha, \beta, \gamma, \delta) = cyc = \gamma + \delta + \begin{cases} \lfloor \frac{\delta}{\beta} \rfloor (\alpha - \beta) & \text{if } \alpha \leq \beta \\ -\lfloor \frac{\gamma}{\alpha} \rfloor (\alpha - \beta) & \text{if } \alpha > \beta \end{cases}$$

As proved in [13][14], we can compute the parameters α, β, γ and δ of a cycloid from its net presentation from the system parameters τ_0, τ_a, A and *cyc*. τ_0 and τ_a refer to the standard initial marking M_0. τ_0 is the number of transitions having as least one marked input place, τ_a is the number of active transitions,

A is (as before) the number of all transitions and cyc is the minimal length of transition cycles. The corresponding equivalence is denoted as *σ-equivalence*. Similar to the theory of regions, the following procedures do not necessarily give a unique result. But for $\alpha \neq \beta$ the resulting cycloids are isomorphic.

Definition 18. *Cycloid systems with identical system parameters τ_0, τ_a, A and cyc are called σ-equivalent.*

Theorem 19 ([13][14]). *Given a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ in its net representation (S, T, F, M_0) where the parameters τ_0, τ_a, A and cyc are known (but the parameters $\alpha, \beta, \gamma, \delta$ are not). Then a σ-equivalent cycloid $\mathcal{C}(\alpha', \beta', \gamma', \delta')$ can be computed by $\alpha' = \tau_0$, $\beta' = \tau_a$ and for γ', δ' by some positive integer solutions of the following formulas using these settings of α and β:*

a) case $\alpha' > \beta'$: $\gamma' \ \mathbf{mod}\ \alpha' = \frac{\alpha' \cdot cyc - A}{\alpha' - \beta'}$ and $\delta' = \frac{1}{\alpha'}(A - \beta' \cdot \gamma')$,
b) case $\alpha' < \beta'$: $\delta' \ \mathbf{mod}\ \beta' = \frac{\beta' \cdot cyc - A}{\beta' - \alpha'}$ and $\gamma' = \frac{1}{\beta'}(A - \alpha' \cdot \delta')$,
c) case $\alpha' = \beta'$: $\gamma' = \lceil \frac{cyc}{2} \rceil$ and $\delta' = \lfloor \frac{cyc}{2} \rfloor$

These equations may result in different cycloids which are isomorphic in the cases a) and b).

Definition 20. *Given a net $\mathcal{N} = (S, T, F), t \in T, M \subseteq S$, Petri defines [8]:*

a) $Contact(t, M) :\Leftrightarrow\ {}^{\bullet}t \subseteq M \ \wedge\ t^{\bullet} \cap M \neq \emptyset$
b) $ReverseContact(t, M) :\Leftrightarrow\ t^{\bullet} \subseteq M \ \wedge\ {}^{\bullet}t \cap M \neq \emptyset$
c) $Transjunction(t, M) :\Leftrightarrow\ {}^{\bullet}t \cap M \neq \emptyset \ \wedge\ t^{\bullet} \cap M \neq \emptyset$

A net system $(\mathcal{N}, M_0)$ is

d) safe *if $\forall M \in \mathcal{R}(\mathcal{N}, M_0)\ \forall t \in T : \neg Contact(t, M) \wedge \neg ReverseContact(t, M)$,*
e) secure *if it is safe and $\forall M \in \mathcal{R}(\mathcal{N}, M_0)\ \forall t \in T : \neg Transjunction(t, M)$ and*
f) live[3] *if $\forall M \in \mathcal{R}(\mathcal{N}, M_0)\ \forall t \in T\ \exists M' \in \mathcal{R}(\mathcal{N}, M_0) : M \xrightarrow{*} M' \ \wedge t$ active in M'.*

Theorem 21 ([14]). *The net system $(\mathcal{N}, M_0) = (S, T, F, M_0)$ of a cycloid system $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ is live and safe and if $\gamma, \delta \geq 2$ it is also secure.*

Circular traffic queues are composed by a number of c sequential and interacting processes of equal length. In the formalism of cycloids this corresponds to a number of β disjunct processes of equal length p. Cycloids with such a property are called *regular*.

Definition 22. *A cycloid $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ with area A is called* regular *if for each $\eta \in \{0, \cdots, 1 - \beta\}$ the set set $\{t_{\xi,\eta} | 1 \leq \xi \leq p\}$ with $p = \frac{A}{\beta}$ of transitions together with the places within forms an elementary cycle[4] and all these sets are disjoint. p is called the process length of the regular cycloid. A regular cycloid together with its regular initial marking M_0 is called a regular cycloid system $(\mathcal{C}, M_0)$.*

[3] *live* is used in the usual form (e.g. see [4], page 59)
[4] An elementary cycle is a cycle where all nodes are different.

Theorem 23. *A cycloid $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ is regular if and only if β is a factor of δ.*

Proof. We first prove that starting in any point $t_{\xi,\eta}$ of the fundamental parallelogram of the cycloid and proceeding in direction of the ξ-axis we will return to $t_{\xi,\eta}$ after passing $p=\frac{A}{\beta}$ transitions.

By Theorem 12, to decide $t_{\xi+p,\eta}\equiv t_{\xi,\eta}$ it is sufficent to check whether $\pi(\mathbf{v})=\pi(((\xi+p),\eta)-(\xi,\eta))\in\mathbb{Z}\times\mathbb{Z}$. Therefore we compute $\pi(\mathbf{v})=\pi(\begin{pmatrix}p\\0\end{pmatrix})=\frac{1}{A}\begin{pmatrix}\delta & -\gamma\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}p\\0\end{pmatrix}=\begin{pmatrix}\frac{\delta\cdot p}{A}\\ \frac{\beta\cdot p}{A}\end{pmatrix}$. $\frac{A}{\beta}$ is the smallest value for p to obtain an integer value in the second component of the last vector. Therefore an equivalent point is not reached before passing p transitions and the cycle is elementary. By the first component, to fulfill $\frac{\delta\cdot p}{A}=\frac{\delta}{\beta}\in\mathbb{Z}$ it is necessary and sufficient that β is a factor of δ. It follows that there is a number of β such elementary cycles of length $\frac{A}{\beta}$ covering the entire set of A transitions. Therefore no pair of these cycles can have a common transition. $\square$

Regular cycloids can be seen as a system of β disjoint sequential, but cooperating processes. To exploit this structure we define specific coordinates, called *regular coordinates*, which are adapted to the process structure. The process of a traffic item a_1 starts with transition $t_{0,0}$ which is denoted $[t_1,a_1]$, having the input place $[s_0,a_1]$. The next transitions are $[t_2,a_1]$ up to $[t_p,a_1]$ and then returning to $[t_1,a_1]$. The other processes for a_2 to a_c (with $\beta=c$) are denoted in the same way. In Figure 5 these transitions are shown together with the coordinates of the Petri space. The latter will be also called standard coordinates. Not all places and arrows are represented in this figure.

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Definition 24. *Given a regular cycloid $\mathcal{C}(\alpha,\beta,\gamma,\delta)$* regular coordinates *are defined as follows: transitions of process $j\in\{1,\cdots,\beta\}$ each with length p are denoted by $\{[t_1,a_j],\cdots,[t_p,a_j]\}$. For each transition $[t_i,a_j]$ the output places are called $[s_i,a_j]$ and $[s'_i,a_j]$. The output transition of $[s_i,a_j], 1\leq j\leq\beta$ is $[t_{(i+1)\,mod\,p},a_j]$. The output transition of $[s'_i,a_j]$ follows from the definition of the cycloid and is given in Lemma 25. Regular coordinates are related to standard coordinates of the Petri space by defining the following initial condition $[t_1,a_j]:=t_{1-j,1-j}$ for $1\leq j\leq\beta$.*

While the output place $[s'_i,a_j]$ in regular coordinates takes its name from the input transition, it remains to determine the output transition according to the corresponding standard coordinates.

Lemma 25. *The injective mapping* stand *from regular to standard coordinates is given by $stand([t_i,a_j])=t_{i-j,1-j}$ for $1\leq i\leq p$ and $1\leq j\leq\beta$. The output transition of $[s'_i,a_1]$ is $[s'_i,a_1]^\bullet=[t_{(i+\alpha+\beta-1)\,mod\,p},a_\beta]$ while for $1<j\leq\beta$ the output transition of $[s'_i,a_j]$ is $[t_{(i-1)\,mod\,p},a_{j-1}]$. If $\beta=\gamma=\delta$ the two cases coincide.*

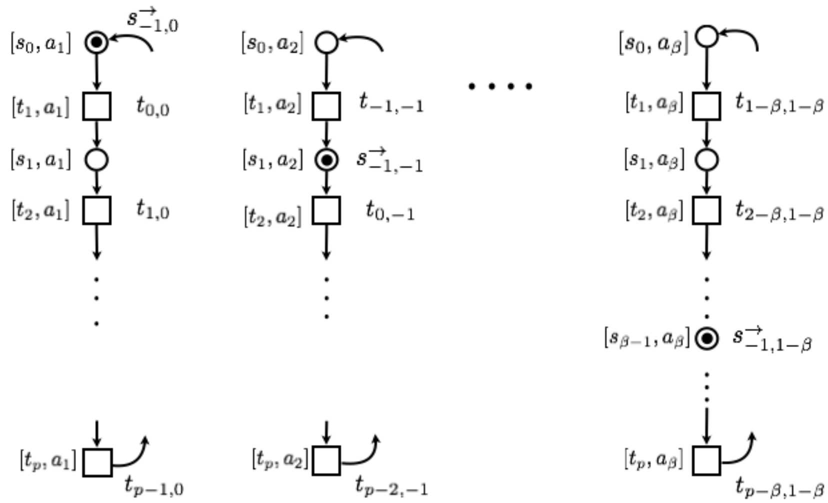

Fig. 5. Regular cycloid system with regular and standard coordinates (extract).

Proof. For a given j by Definition 24 we have $[t_1, a_j] := t_{1-j,1-j}$. Adding a value $i-1$ to the index of t_1 we obtain the index of t_i, hence $stand([t_i, a_j]) := t_{1-j+(i-1),1-j}$.

To prove $[s'_i, a_j]^\bullet = [t_{(i-1)\,mod\,p}, a_{j-1}]$ for $1 < j \leq \beta$ we first compute the corresponding standard coordinate $stand([t_i, a_j]) = t_{i-j,1-j}$. To obtain the output transition of $[s'_i, a_j]$ we go to the next transition in η-direction $t_{i-j,1-j+1}$ and obtain $stand^{-1}(t_{i-j,2-j}) = [t_{(i-j)-(2-j)+1}, a_{1-(2-j)}] = [t_{i-1}, a_{j-1}]$, where $mod\,p$ is omitted.

To make the proof for $[s'_i, a_1]$ we start with $[t_i, a_1]$ and compute again $stand([t_i, a_1]) = t_{i-1,0}$. The next transition in η-direction is $t_{i-1,1}$ (see Figure 6). Using Theorem 12 we prove the equivalence $t_{i-1,1} \equiv t_{i-1+\alpha,1-\beta}$:
$$\pi(\begin{pmatrix} i-1 \\ 1 \end{pmatrix} - \begin{pmatrix} i-1+\alpha \\ 1-\beta \end{pmatrix}) = \pi(\begin{pmatrix} -\alpha \\ \beta \end{pmatrix}) = \frac{1}{A}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} -\alpha \\ \beta \end{pmatrix} = \frac{1}{A}\begin{pmatrix} -\delta\cdot\alpha - \gamma\cdot\beta \\ -\beta\cdot\alpha + \alpha\cdot\beta \end{pmatrix} = \frac{1}{A}\begin{pmatrix} -A \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in \mathbb{Z}\times\mathbb{Z}$$

Going back to the corresponding regular coordinates the desired result is obtained: $stand^{-1}(t_{i-1+\alpha,1-\beta}) = [t_{(i-1+\alpha)-(1-\beta)-1}, a_{1-(1-\beta)}] = [t_{i+\alpha+\beta-1}, a_\beta)]$ where $mod\,p$ is omitted again. If $\beta = \gamma = \delta$ the two cases coincide as $p = \alpha + \beta$. □

Corollary 26. *The regular initial marking of a regular cycloid system* $\mathcal{C}(\alpha, \beta, \gamma, \delta, M_0)$ *with process length* p *is*

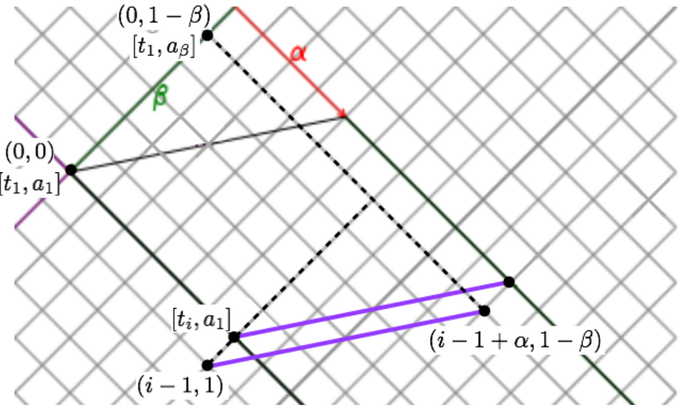

Fig. 6. Computation of the input transition of $[s'_i, a_1]$.

$M_0 = \{[s_i, a_{(i+1)\,mod\,\beta}] | 0 \leq i \leq \beta - 1\} \cup \{[s'_i, a_1] | p - \alpha + 1 \leq i \leq p\}$.
For later reference, we note $\overleftarrow{{}^{\bullet}}[t_\beta, a_\beta] = [s_{p-\alpha+1}, a_1]$.

Proof. Since the mapping *stand* is defined on transitions, from the first place of M_0 in Corollary 26 $[s_0, a_{(0+1)\,mod\,\beta}]$ we go to its output transition $[t_1, a_1]$ and apply $stand([t_1, a_1]) = t_{0,0}$. Going back in ξ-direction we obtain $s^{\rightarrow}_{-1,0}$, which is the first element in Definition 16. Doing the same with $[s_{\beta-1}, a_{\beta\,mod\,\beta}]$ we come via $stand([t_\beta, a_\beta]) = t_{\beta-\beta,1-\beta} = t_{0,1-\beta}$ to $s^{\rightarrow}_{-1,1-\beta}$. Hence we obtain the entire forward places from Definition 16. Since the mapping *stand* is injective we can conclude also in the inverse direction.

To prove the second part of the union recall that the last traffic item a_β is activated. Therefore also the backward input place $[s'_i, a_1]$ of $[t_\beta, a_\beta]$ must be marked. Using Lemma 25 the value of i must satisfy $[s'_i, a_1]^{\bullet} = [t_{(i+\alpha+\beta-1)\,mod\,p}, a_\beta] = [t_\beta, a_\beta]$, hence $(i + \alpha + \beta - 1)\,mod\,p = \beta$ and $i = (1 - \alpha)\,mod\,p = p - \alpha + 1$. This holds since $\beta | \delta \Rightarrow \beta \leq \delta$ and therefore $p = \frac{A}{\beta} = \alpha\frac{\delta}{\beta} + \gamma > \alpha$. The marked place in question is therefore $[s'_i, a_1] = [s'_{p-\alpha+1}, a_1]$. To determine the other elements of $\{[s'_i, a_1] | p - \alpha + 1 \leq i \leq p\}$ recall that the last traffic item a_β should be able to make α steps before any other transition has to occur. Therefore also the places $[s'_{p-\alpha+2}, a_1]$ to $[s'_{p-\alpha+\alpha}, a_1]$ must be marked in the regular initial marking. □

As an example see the regular cycloid system $\mathcal{C}(3, 3, 3, 3, M_0)$ in Figure 10. The given regular initial marking is $\{[s_0, a_1], [s_1, a_2], [s_2, a_3], [s'_4, a_1], [s'_5, a_1],$ $[s'_6, a_1], [s'_7, a_1]\}$. By bold circles the standard initial marking is also given.

It is useful in some cases to express the minimal cycle length *cyc* of a regular cycloid by its process length p. While this can be perfectly done for the case $\alpha \leq \beta$ in the complementary case only partial results are achieved. These cover, however, all the cases required in Section 4. Compared with general cycloids the lack of symmetry of regular cycloids becomes apparent by these results.

Lemma 27. *Let be $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ a regular cycloid with process length p and minimal cycle length cyc.*

a) $cyc = p$ if $\alpha \le \beta$.
b) If $\alpha \ge \beta$ and α divides γ then $cyc = \frac{\beta}{\alpha} \cdot p$.
c) $cyc = 2 \cdot \beta$ if $\alpha \ge \beta = \gamma = \delta$.

Proof. a) By Theorem 17 and $\delta = m \cdot \beta$ for some $m \in \mathbb{Z}$ we obtain $cyc = \gamma + \delta + \lfloor \frac{\delta}{\beta} \rfloor \cdot (\alpha - \beta) = \gamma + m \cdot \beta + \lfloor \frac{m \cdot \beta}{\beta} \rfloor \cdot (\alpha - \beta) = \gamma + m \cdot \alpha$. This term equals $p = \frac{A}{\beta} = \frac{1}{\beta}(\alpha\delta + \beta\gamma) = \frac{1}{\beta}(\alpha \cdot m \cdot \beta + \beta\gamma) = \alpha \cdot m + \gamma$.

b) If $\alpha \ge \beta$ case a) applies to the dual cycloid $\mathcal{C}(\beta,\alpha,\delta,\gamma)$ (Definition 14), which is regular since α divides γ. Hence $cyc = p'$ where $p' = \frac{A}{\alpha} = \frac{\beta \cdot p}{\alpha} =$ is the process length of the dual cycloid which is isomorphic to $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ by Theorem 14.

c) If $\alpha > \beta = \gamma = \delta$ then $cyc = \gamma + \delta - \lfloor \frac{\gamma}{\alpha} \rfloor \cdot (\alpha - \beta) = \beta + \beta - 0 \cdot (\alpha - \beta)$. If $\alpha = \beta = \gamma = \delta$ then $cyc = \gamma + \delta + \lfloor \frac{\delta}{\beta} \rfloor \cdot (\alpha - \beta) = \beta + \beta + 1 \cdot 0$. $\square$

The regular cycloid $\mathcal{C}(4,3,3,6)$ does not satisfy any of the conditions of Theorem 27. The parameters in question are $cyc = 9$, $p = 11$ and $\frac{\beta}{\alpha} \cdot p = \frac{3}{4} \cdot 11$.

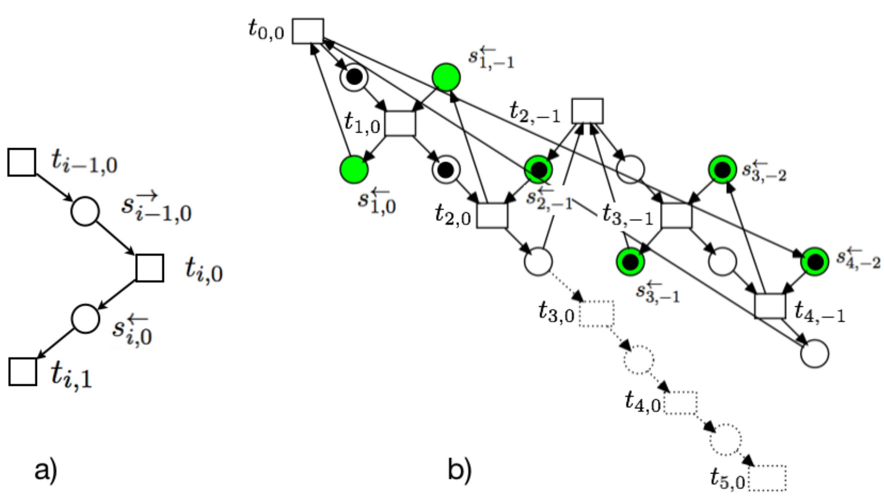

Fig. 7. Cycloid $\mathcal{C}(4,2,1,1)$ with regular initial marking and minimal cycle in a)

4 Net Representations of Traffic Queues

In this section different forms of traffic queues are represented by nets and cycloids. We first show that the basic net for a circular traffic queue in Figure 19 a) equals $\mathcal{C}(g, c, 1, 1)$. The proof starts from the specification from Definition 7 to derive the parameters τ_0, τ_a, A and cyc. Then using Theorem 19 the cycloid is determined in terms of the parameters α, β, γ and δ. This approach has the advantage, that it is undoubtedly to obtain a cycloid as a result. This is, however, not a complete proof, since it was not shown, that the specifications are correct and complete. Therefore a proof is added showing that the nets are isomorphic. This proof is complete but does not give insight into structural properties. Afterwards similar proofs are given for more complex cycloids.

4.1 Representation of a circular queues by safe cycloids

Using Theorem 19 cycloids can be generated directly from Theorem 5 and the specifications of a circular traffic queue in Definition 7.

Theorem 28. *Searching for a safe net meeting the specifications of a circular traffic queue with gaps tq-*$1(c, g)$ *(Definition 6) the cycloid* $\mathcal{C}_0(g, c) := \mathcal{C}(g, c, 1, 1)$ *can be deduced. More precisely, the cycloid is isomorphic to the net* $\mathcal{N}_{basic}(c, g)$ *of Figure 19 a).*

Proof. From Petri's papers [8], [7] or [10] it follows definitely that the first two parameters are $\beta = c$ and $\alpha = g$. To follow a more formal approach we determine the parameters τ_0, τ_a, A and cyc and then apply Theorem 19 to determine the parameters α, β, γ and δ of the cycloid $\mathcal{C}(g, c, 1, 1)$.

Consider first the case $g \geq c$ and the state $a_1 \times^{k_1} a_2 \times^{k_2} \cdots a_c \times^{k_c}$. Due to the assumption $g \geq c$ it is possible to have $k_j \geq 1 \ \ (1 \leq j \leq c)$. This means that all traffic items a_j are able to move right, i.e they are active and therefore $\tau_a = \beta = c$. τ_0 is the number of initially marked transitions. All g positions of the queue which are empty send a release message to the left. Therefore a number of $\tau_0 = \alpha = g$ transitions are marked, among these the c active transitions mentioned before. Given a fixed traffic item a and a fixed position in the queue, in a recurrent sequence the traffic item enters the position exactly once. Hence there are $g + c$ transitions. In the same step the traffic item a gives a release message to enable the access to the position it is leaving. This results in a minimal cycle of length 2.

With the parameters $\tau_a = \beta = c, \tau_0 = \alpha = g, A = g+c$ and $cyc = 2$ obtained, we compute with Theorem 19 : $\gamma \ mod \ g = \frac{\alpha \cdot cyc - A}{\alpha - \beta} = \frac{g \cdot 2-(g+c)}{g-c} = \frac{g-c}{g-c} = 1$. With a solution $\gamma = 1$ of this equation we obtain $\delta = \frac{1}{\alpha}(A - \beta \cdot \gamma) = \frac{1}{g}(c+g-c \cdot 1) = 1$.

Next we prove that this cycloid $\mathcal{C}(g, c, 1, 1)$ is isomorphic to the basic tq-net $\mathcal{N}_{basic}(c, g)$ from Definition 32 and Figure 19 a). Starting in the origin $(0, 0)$ of the fundamental parallelogram of $\mathcal{C}(g, c, 1, 1)$ using Theorem 12 we compute the smallest point $(\xi, 0)$ on the ξ-axis equivalent to $(0, 0)$:

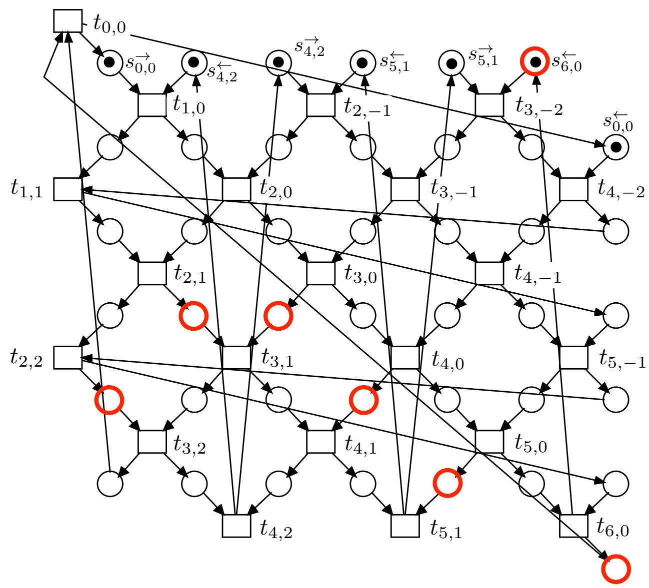

Fig. 8. Cycloid $\mathcal{C}(4,3,3,3)$ represented as a net system.

$$\begin{pmatrix} m \\ n \end{pmatrix} = \tfrac{1}{A}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}\mathbf{v} = \tfrac{1}{g+c}\begin{pmatrix} 1 & -1 \\ c & g \end{pmatrix}\begin{pmatrix} \xi \\ 0 \end{pmatrix} = \tfrac{1}{g+c}\begin{pmatrix} \xi \\ \xi \cdot c \end{pmatrix}.$$

The smallest positive integer for m is $\xi = g + c$. Since $A = g + c$ all transitions have their position on the ξ-axis from $(0,0)$ to $(g+c-1,0)$ and form a cycle. This cycle is isomorphic to the cycle $t_1 \cdots t_{c+g}$ the basic net of Figure 19 a). It remains to prove that $t_{i,0}$ $(1 \leq i \leq c+g)$ is connected to $t_{i-1,0}$ via a place isomorphic to s'_i forming a cycle of length 2. As the backward output place of $t_{i,0}$ is $s^{\leftarrow}_{i,0}$ and the output transition of the latter is $t_{i,1}$ (see Figure 7 a)) we have to prove: $t_{i-1,0} \equiv t_{i,1}$. This is done by Theorem 12 using $\mathbf{v} = (i,1) - (i-1,0)$:

$$\begin{pmatrix} m \\ n \end{pmatrix} = \tfrac{1}{A}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}\mathbf{v} = \tfrac{1}{g+c}\begin{pmatrix} 1 & -1 \\ c & g \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{1}{g+c}\begin{pmatrix} 0 \\ c+g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z} \times \mathbb{Z}.$$

For the case $g < c$ we observe that the net from Figure 19 a) is symmetric in the following sense. Interpreting the places $s'_1, s'_2, \cdots s'_{c+g}$ to be the slots of the traffic items instead of $s_1, s'_2, \cdots s_{c+g}$ we obtain an isomorphic system. By the construction in the first part of this proof we obtain the cycloid $\mathcal{C}(c,g,1,1)$ which is isomorphic to $\mathcal{C}(g,c,1,1)$ by Theorem 14. Therefore the theorem holds also in this case. □

As an example, in Figure 7 b) the cycloid $\mathcal{C}(4,2,1,1)$ is shown. To illustrate the preceding proof by dashed lines the following equivalent transitions are given: $t_{3,0} \equiv t_{2,-1}$, $t_{4,0} \equiv t_{3,-1}$ and $t_{5,0} \equiv t_{4,-1}$. The transitions $t_{0,0}, \cdots, t_{5,0}$ on the ξ-axis are instances of the transitions $t_{0,0}, \cdots, t_{g+c-1,0}$ in the proof. The places $s^{\leftarrow}_{1,0}$, $s^{\leftarrow}_{1,-1}$, $s^{\leftarrow}_{2,-1}$, $s^{\leftarrow}_{3,-1}$, $s^{\leftarrow}_{3,-2}$ and $s^{\leftarrow}_{4,-2}$ correspond to the "complementary places" $s'_1, s'_2, \cdots, s'_{c+g}$ of the net from Figure 19.

4.2 Representation of circular queues by secure cycloids

Cycloids are more expressive to model circular traffic queues and of particular interest with respect to different questions. For instance, they respect individual trafic items, similar to coloured nets. To distinguish them from the preceeding model in Theorem 28 we assume $c > 1$ to obtain secure cycloids.

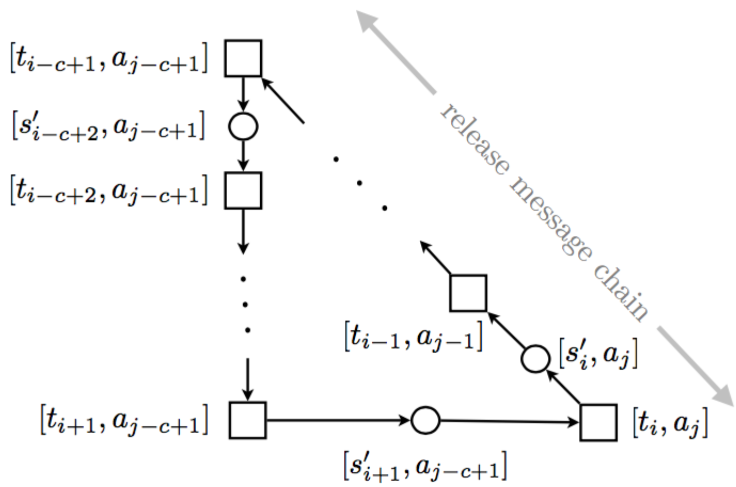

Fig. 9. Release message cycle in $\mathcal{C}(g, c, c, c)$.

Most important in the next theorem is the computation of the minimal length *cyc* of cycles. Candidates are the length of the processes $p = c + g$ and of the *release message cycle* $rm = 2 \cdot c$. The latter starts with the *release message chain* which is the sequence of release messages from traffic item a_c to a_{c-1}, from a_{c-1} to a_{c-2} and so on. This can be seen as the most important synchronization mechanism of the system. Finally we obtain $cyc = min\{p, rm\}$.

Theorem 29. *Searching for a regular cycloid meeting the specifications of a circular traffic queue with gaps tq-1(c, g) (Definition 6) the cycloid $\mathcal{C}_1(g, c) := \mathcal{C}(g, c, c, c)$ can be deduced. It has the process length $p = g + c$ and minimal cycle length* $cyc = \begin{cases} p = g + c & \text{if } c > g \\ 2 \cdot c & \text{if } g \geq c \end{cases}$ *(in accordance with Lemma 27 a) and c)).*

Proof. For the determination of α and β we argue as in Theorem 28, i.e. $\alpha = g$ and $\beta = c$. If the total number of transitions is A and there are c traffic items with the same process length, this process length is $p = \frac{A}{c}$. In a secure cycloid, due to the Lemma of Petri/Stehr, all these cycles are disjoint. Petri made the assertion [10] that a cycloid with $\gamma, \delta \geq 2$ is secure, if every pair of successive arcs lies on a basic cycle. A basic circuit is a cycle with exactly one edge marked. This has been proved (also for the more general case of T-systems) by Stehr [12] and it is therefore called the Lemma of Petri/Stehr.

As there is a number c of disjoint communicating processes of equal length p the net to be constructed is a regular cycloid (Definition 22) and we can use the naming of Definition 24 and Figure 5. The missing places of this net are obtained by considering condition b) of the specification in Definition 7 as follows. Transition $[t_i, a_j]$ models the step of car a_j in position i. In this step it is sending a release message to be received by car $a_{(j-1)\, mod\, n}$ by transition $[t_{(i-1)\, mod\, n}, a_{(j-1)\, mod\, c}]$. This is modelled by a new place $[s'_i, a_j]$ and arrows $([t_i, a_j], [s'_i, a_j]$ and $([s'_i, a_j], [t_{(i-1)\, mod\, n}, a_{(j-1)\, mod\, c}]$ (see Figure 14). As a result, we obtain additional $c \cdot p$ places, i.e. in total the double of the number of transitions.

To find a minimum length cycle, we next consider a sequence of transitions, which we call the *release message chain*, *rm-chain* for short. The rm-chain starts in some transition $[t_i, a_j]$ and continues via place $[s'_i, a_j]$ and transition $[t_{i-1}, a_{j-1}]$ down to $[t_{i-c+1}, a_{j-c+1}]$, i.e. the process of all traffic items are passed (see Figure 9). Again, first and second index is computed modulo n and c, respectively. In the following the rm-chain is closed to a *release message cycle*.

To close the cycle we continue within the process cycle of car a_{j-c+1} a number of c transitions until $[t_{i-c+1+c}, a_{j-c+1}] = [t_{i+1}, a_{j-c+1}]$ and $[t_{i+1}, a_{j-c+1}]^{\overleftarrow{\bullet}} = [s'_{i+1}, a_{j-c+1}]$. For the final step by Lemma 25 there are two cases to compute $[s'_{i+1}, a_{j-c+1}]^{\bullet} = [t_i, a_j]$ (Figure 9). If $a_{j-c+1} = a_1$ then $[s'_{i+1}, a_{j-c+1}]^{\bullet} = [t_{i+1+g+c-1}, a_c] = [t_i, a_j]$ since $(i+1+g+c-1)\, mod\, (g+c) = i$ and $j-c+1 = 1 \;\Rightarrow\; c = j$. If $a_{j-c+1} \neq a_1$ then $[s'_{i+1}, a_{j-c+1}]^{\bullet} = [t_i, a_{j-c+1-1}] = [t_i, a_j]$ since $(j-c)\, mod\, c = j$.

The length of the rm-cycle is $rm = 2 \cdot c$ transitions and there is non shorter cycle connecting the process cycles. These process cycles have the length of $p = \frac{A}{c} = \frac{c \cdot (c+g)}{c} = c+g$ transitions and are also candidates for an overall minimal cycle. Therefore the overall minimal cycle of the cycloid is $cyc = min\{2 \cdot c, c+g\}$ or $cyc = \begin{cases} 2 \cdot c & \text{if } g \geq c \\ c+g & \text{if } c > g \end{cases}$

For a regular cycloid we construct the following recurrent transition sequence, starting with $[t_c, a_c]$ in the standard initial marking: $[t_c, a_c], [t_{c-1}, a_{c-1}], \cdots,$ $[t_1, a_1], \cdots\cdots,\ [t_{c+n-1}, a_c], [t_{c+n}, a_{c-1}], \cdots][t_n, a_1]$ of length $c \cdot n$. By [14] the cycloid is strongly connected and therefore has a T-invariant of the form $(x, x, \cdots, x), x \in \mathbb{Z}$ (see reference [3]). This means that all recurrent sequences contain each transition exactly once, as it is in the constructed one and we have $A = \mathit{\Gamma}(c, g) = c \cdot (g+c)$ (by Definition 7 c), Theorem 5).

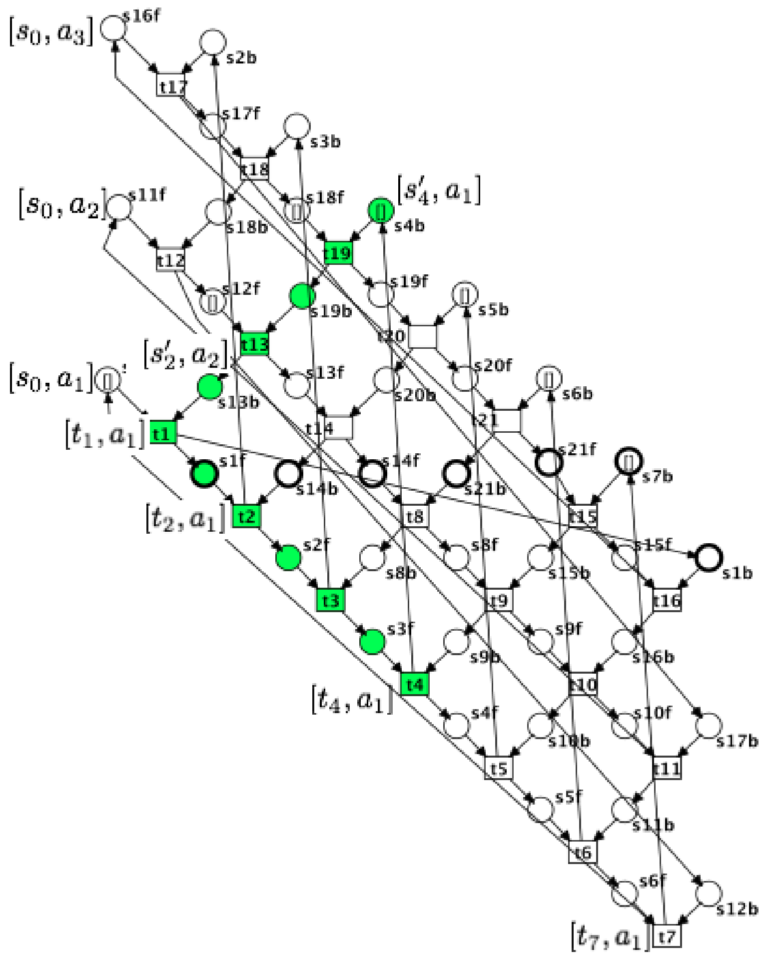

Fig. 10. $\mathcal{C}(4, 3, 3, 3)$ represented as regular cycloid.

Using this value of cyc, and the parameters $\alpha = g$, $\beta = c$ and A we can apply Theorem 19:

case: $g = \alpha > \beta = c$: γ **mod** $g = \frac{\alpha \cdot cyc - A}{\alpha - \beta} = \frac{g \cdot 2 \cdot c - c \cdot (g+c)}{g-c} = c$, where $\gamma = c$ is a solution and $\delta = \frac{1}{\alpha}(A - \beta \cdot \gamma) = \frac{1}{g}(c \cdot (g+c) - c \cdot c) = c$.

case: $g = \alpha < \beta = c$: γ **mod** $c = \frac{\beta \cdot cyc - A}{\alpha - \beta} = \frac{g \cdot 2 \cdot c - c \cdot (g+c)}{g-c} = c$, where $\gamma = c$ is a solution and $\delta = \frac{1}{\alpha}(A - \beta \cdot \gamma) = \frac{1}{g}(c \cdot (g+c) - c \cdot c) = c$.

case: $g = \alpha = \beta = c$: $\gamma = \delta = \lfloor \frac{cyc}{2} \rfloor = c$. □

The cycloid $\mathcal{C}(4,3,3,3)$ from Figure 8 is an example for Theorem 29. In Figure 10 the same cycloid is given but its graph is adapted to the definition of a regular cycloid. Some regular coordinates, like $[s_0, a_1]$ and $[t_1, a_1]$, are added. One of the rm-cycles is highlighted. It has the length $rm = 2 \cdot c = 2 \cdot \beta = 6$.

Also in the next theorem the length rm of the release message cycle is important to determine cyc. Here we will obtain $rm = p + (c - g)$ and $cyc = min\{p, rm\} = p$ if $c \geq g$. The case $c < g$ is solved by symmetry considerations.

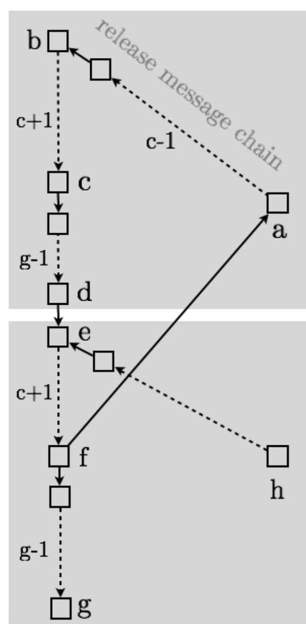

Fig. 11. The release message cycle in $\mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta})$ with $\frac{g}{\Delta} = 2$.

Theorem 30. *Searching for a regular cycloid meeting the specifications of a circular traffic queue tq-*$2(c, g)$ *(Definition 6) the cycloid* $\mathcal{C}_2(g, c) := \mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta})$

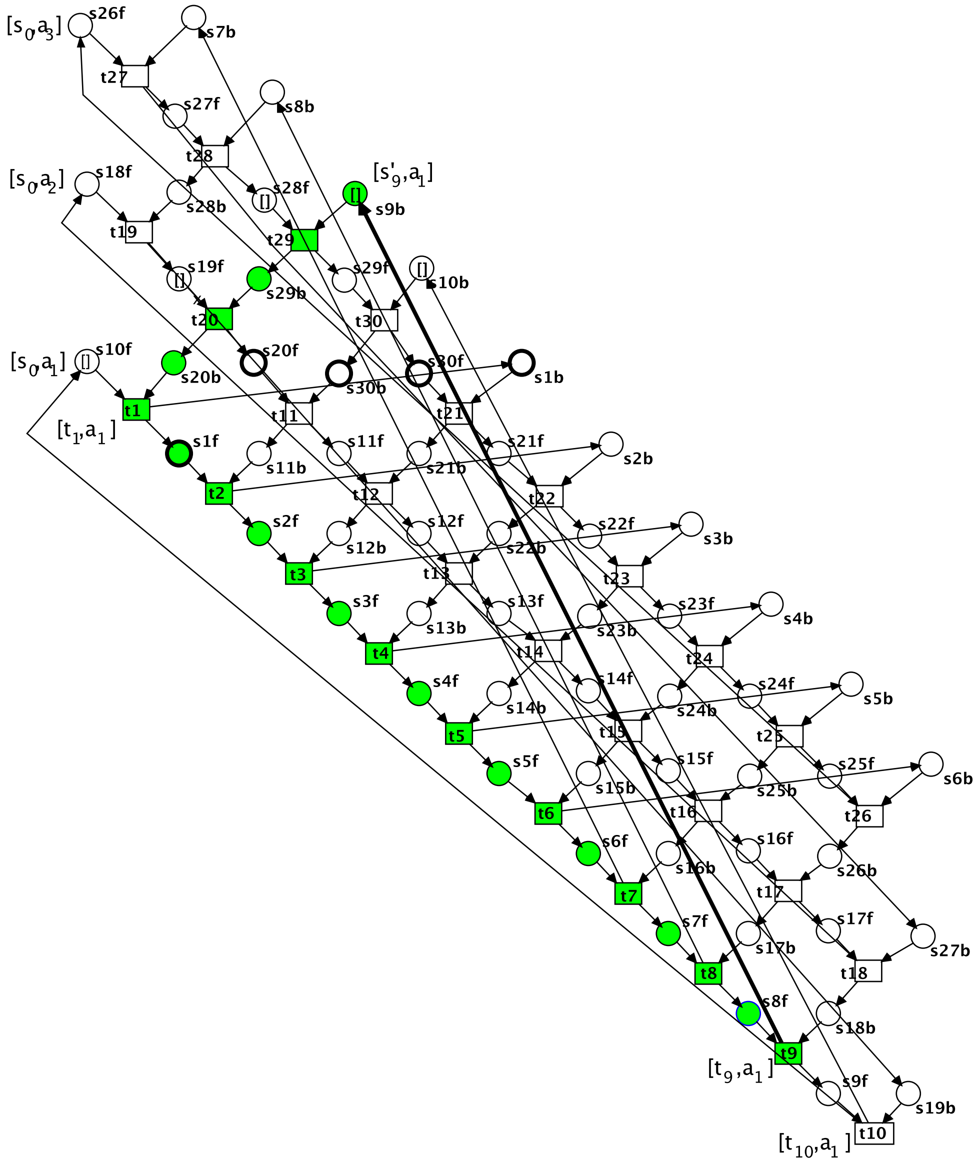

Fig. 12. $\mathcal{C}(2,3,6,6)$ represented as a regular cycloid.

can be deduced. It has a process length $p = \frac{g}{\Delta}(g+c)$ *and minimal cycle length* $cyc = \begin{cases} p = \frac{g}{\Delta}(g+c) & \textit{if } c > g \\ \frac{c}{g} \cdot p = \frac{c}{\Delta}(g+c) & \textit{if } g \geq c \end{cases}$ *(in accordance with Lemma 27 a) and b)).*

Proof. The proof is the same as for Theorem 29 until the computation of the length of the rm-chain. Due to the increased length of processes the rm-chain cannot be closed as in this case. By the specification of Definition 7 the transition system of tq-2(c, g) is a composition of $\frac{g}{\Delta}$ copies of the transition system of tq-1(c, g). This is schematically shown in Figure 11 for the case $\frac{g}{\Delta} = 2$. At full arcs one place is omitted. At dotted arcs with label λ a number of λ transitions is supposed (including beginning and end of the arc). For the upper copy the the rm-chain from transition a via transitions b and c to d cannot be closed, but is continued to the lover copy. To this end a number $g-1$ of additional transitions are to be passed. From e via f the rm-chain is closed to complete a rm-cycle. In the general case the structure of the upper copy is repeated a number of $\frac{g}{\Delta}$ times. Hence, for the overall length of the rm-cycle is the number of transitions on a path from transition a to transition f by repeating the path from b to d a number of $\frac{g}{\Delta} - 1$ times: $rm = (c-1) + (\frac{g}{\Delta} - 1) \cdot (c+1+g-1) + c + 1 = \frac{g}{\Delta} \cdot (g+c) + (c-g) = p + (c-g)$. If $c \geq g$ the length of the rm-cycle is not smaller than the process-length p, which is the minimal cycle in this case, hence $cyc = \frac{g}{\Delta} \cdot (g+c)$. The model tq-2(c, g) is symmetric with respect to c and g. The same proof can be made with c and g interchanged and $cyc = \frac{c}{\Delta} \cdot (g+c)$ if $c \leq g$. For the value of $A = \Xi(c, g)$ we argue as in the proof of Theorem 29.

Using these values of cyc, and the parameters $\alpha = g$, $\beta = c$ and $A = \Xi(c, g) = \frac{g}{\Delta} \cdot (g+c) \cdot c$ we can apply Theorem 19:

case: $g = \alpha > \beta = c$: $\gamma \textbf{ mod } g = \frac{\alpha \cdot cyc - A}{\alpha - \beta} = \frac{g \cdot \frac{c}{\Delta} \cdot (g+c) - \frac{g}{\Delta} \cdot (g+c) \cdot c}{g-c} = 0$, where $\gamma = g$ is a solution and $\delta = \frac{1}{\alpha}(A - \beta \cdot \gamma) = \frac{1}{g}(\frac{g}{\Delta}(g+c) \cdot c - c \cdot g) = c \cdot (\frac{g+c}{\Delta} - 1)$. The resulting cycloid is $\mathcal{C}(g, c, g, c \cdot (\frac{g+c}{\Delta} - 1))$. By Theorem 14 c) this cycloid is equivalent to $\mathcal{C}(\alpha, \beta, \gamma + x \cdot \alpha, \delta - x \cdot \beta)$. With $x = \frac{c}{\Delta} - 1$ we obtain $\gamma' = \gamma + x \cdot \alpha = g + (\frac{c}{\Delta} - 1) \cdot g = \frac{g \cdot c}{\Delta}$ and $\delta' = \delta - x \cdot \beta = c \cdot (\frac{g+c}{\Delta} - 1) - (\frac{c}{\Delta} - 1) \cdot c = \frac{g \cdot c}{\Delta}$.

case: $g = \alpha < \beta = c$: $\delta \textbf{ mod } c = \frac{\beta \cdot cyc - A}{\alpha - \beta} = \frac{c \cdot \frac{g}{\Delta} \cdot (g+c) - \frac{g}{\Delta} \cdot (g+c) \cdot c}{g-c} = 0$, , where $\delta = c$ is a solution and $\gamma = \frac{1}{\beta}(A - \alpha \cdot \delta) = \frac{1}{g}(\frac{g}{\Delta}(g+c) \cdot c - c \cdot g) = g \cdot (\frac{g+c}{\Delta} - 1)$. The resulting cycloid is $\mathcal{C}(g, c, g \cdot (\frac{g+c}{\Delta} - 1), c)$. By Theorem 14 b) this cycloid is equivalent to $\mathcal{C}(\alpha, \beta, \gamma - x \cdot \alpha, \delta + x \cdot \beta)$. With $x = \frac{g}{\Delta} - 1$ we obtain $\gamma' = \gamma - x \cdot \alpha = g \cdot (\frac{g+c}{\Delta} - 1) - (\frac{g}{\Delta} - 1) \cdot g = \frac{g \cdot c}{\Delta}$ and $\delta' = \delta + x \cdot \beta = c + (\frac{g}{\Delta} - 1) \cdot c = \frac{g \cdot c}{\Delta}$.

case: $g = \alpha = \beta = c$: $\gamma = \delta = \lfloor \frac{cyc}{2} \rfloor = \frac{g \cdot g}{\Delta}$. In all cases we obtain $\mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta})$. □

The cycloid $\mathcal{C}(2, 3, 6, 6)$ from Figure 12 is an example for Theorem 30. Its graph is adapted to the definition of a regular cycloid. Some regular coordinates, like $[s_0, a_1]$ and $[t_1, a_1]$, are added. One of the rm-cycles is highlighted. It has the length $rm = p + (c-g) = 10 + (3-2) = 11$.

4.3 Isomorphisms

While in Section 4.2 the synthesis of cycloids as models of circular traffic queues from the specification has been described the next section is dedicated to analysis. We will prove that the obtained cycloids are behaviour equivalent to the circular traffic queues. This is done using operational semantics, i.e. comparing the transition systems.

Theorem 31. *a) The reachability graph of the cycloid system* $(\mathcal{C}_1(g,c), M_0) := \mathcal{C}(g,c,c,c,M_0)$ *is isomorphic to the labelled transition system* $LTS_p(c,g)$ $(p = n = c+g)$*, where* M_0 *and* ind_0 *are the regular initial marking and state, respectively.*

b) The reachability graph of the cycloid system $(\mathcal{C}_2(g,c), M_0) := \mathcal{C}(g,c,\frac{g\cdot c}{\Delta},\frac{g\cdot c}{\Delta},M_0)$ *is isomorphic to the labelled transition system* $LTS_p(c,g)$ $\;(p = \frac{g}{\Delta}\cdot n = \frac{g}{\Delta}\cdot(c+g))$*, where* M_0 *and* ind_0 *are the regular initial marking and state, respectively.*

In both cases the same holds for the standard initial marking.

Proof. The reachability graph $\mathcal{RG}(\mathcal{N},M_0) = (\mathcal{R}(\mathcal{N},M_0),\rightarrow)$ of the cycloid can be seen as a labelled transition system $LTS' = (States', T', tr', ind'_0)$. Each marking contains the same number $n = g+c$ of marked places as this holds in the initial marking of the T-net. Therefore we also consider such a marking as an ordered set by an index function $ind' : \{1,\cdots,n\} \rightarrow S$, where S is the set of places.

The labelled transition systems $LTS := LTS_p(c,g)$ and LTS' are isomorphic if there are bijective mappings φ and ψ. φ gives for each state $ind \in States$ a corresponding state $ind' = \varphi(ind) \in States'$ and ψ gives for each transition $t \in T$ a corresponding transition $t' = \psi(t) \in T'$. The following condition is required: $(ind_1, t, ind_2) \in tr \;\Leftrightarrow\; (\varphi(ind_1), \psi(t), \varphi(ind_2)) \in tr'$. The cycloids of the theorem are regular by Theorem 23 and we can use its regular coordinates (Definition 24, Lemma 25). In the remaining proof, all indices containing i are understood *modulo* n in a) and *modulo* p in b). All indices containing j are understood *modulo* c.

Part a) of the theorem: As $r = 1$ in this case, we can use C instead of C^r, since 0 is the only possible exponent in a_j^i.

i) **Definition of φ and ψ.**
If $\mathcal{RG}$ and $\mathcal{RG}'$ are the sets of reachable states of LTS and LTS', respectively, then the corresponding states and markings are defined by index functions ind and ind'. Using ind the mapping φ is given by defining the corresponding ind'. Recall that following Definition 24 the places have names of the form $[s_{i-1}, a_j]$ representing traffic item a_j in position i and $[s'_i, a_j]$ representing the release message from transition $[t_i, a_j]$ (Figure 14).
For $1 \leq i \leq n$ let be $ind'(i) := [\hat{s}_i, next(i)]$ with $\hat{s}_i = \begin{cases} s_{i-1} & \text{if } ind(i) \in C \\ s'_i & \text{if } ind(i) \in G \end{cases}$
where $next(i) = a$ if $ind(i) = a \in C$. If $ind(i) = u \in G$ consider, starting in i, the following modulo-n-unfolded sequence of indexed state components of length n:
$ind(i)ind(i+1) \cdots ind(n)ind(1) \cdots ind(i-1)$.
If $u_{i_1} u_{i_2} \cdots u_{i_k} \in G^+$ is a maximal block of elements from G starting in $ind(i) = u = u_{i_1}$ then define: $next(i) := ind((k+1) \bmod n) \in C$.
Hence, if $ind(i) = a_j \in C$ then $next(i) = a_j$ and $ind'(i) = [s_{i-1}, a_j]$. However, if $ind(i) \in G$ then $next(i) = a_j \in C$ which is the first $a_j \in C$ after position i and $ind'(i) = [s'_i, a_j]$.
In the word form a state (as described following Definition 1) can be represented as word of length n in the form $u_1 u_2 \cdots u_r a_1 u_{r+2} \cdots u_{r'} a_2 u_{r'+2} \cdots$ where $u_i = \times$ is in position i. The image by the mapping φ is
$[s'_1, a_1][s'_2, a_1] \cdots [s'_r, a_1][s_r, a_1][s'_{r+2}, a_2] \cdots [s'_{r'}, a_2][s_{r'}, a_2][s'_{r'+2}, a_3] \cdots$. Note that a_1 and a_2 are at positions $r+1$ and $r'+1$, respectively. On top of Figure 13 the case for a state of length $n = 7$ is shown. Below its image under φ is given together with a fragment of the cycloid $\mathcal{C}(4, 3, 3, 3)$.
The mapping $\psi(\langle\langle t_i, a_j \rangle\rangle) = [t_i, a_j]$ ψ is obviously injective and also surjective as $LTS := LTS_p(c, g)$ has a number of $\Gamma(c, g) = c \cdot (c + g)$ transitions (Theorem 5) as well as $(\mathcal{C}(g, c, c, c), M_0)$ with $A = g \cdot c + c \cdot c$.

ii) $(\mathbf{s_1}, \mathbf{t}, \mathbf{s_2}) \in \mathbf{tr} \Rightarrow (\varphi(\mathbf{s_1}), \psi(\mathbf{t}), \varphi(\mathbf{s_2})) \in \mathbf{tr}'$.
A transition from a state component $[i, a_j]$ implies that a traffic item a_j in position i is moving to position $i+1$. This is possible if position $i+1$ is free (see Figure 14). Hence two positions are involved: $ind(i)ind(i+1) = a_j \times$ and this is changed by transition $[t_i, a_j]$ to $ind(i)ind(i+1) = \times a_j$. With the mapping φ, as defined in i), we obtain that $ind'(i)ind'(i+1) = [s_{i-1}, a_j][s'_{i+1}, a_{j+1}]$ is changed to $ind'(i)ind'(i+1) = [s'_i, a_j][s_i, a_j]$. This is exactly the modification performed by the transition $[t_i, a_j]$ in the regular cycloid, as by Definition 24 this transition has the output places $\{[s_i, a_j], [s'_i, a_j]\}$ and by Lemma 25 the input places $\{[s_{i-1}, a_j], [s'_{i+1}, a_{j+1}]\}$.

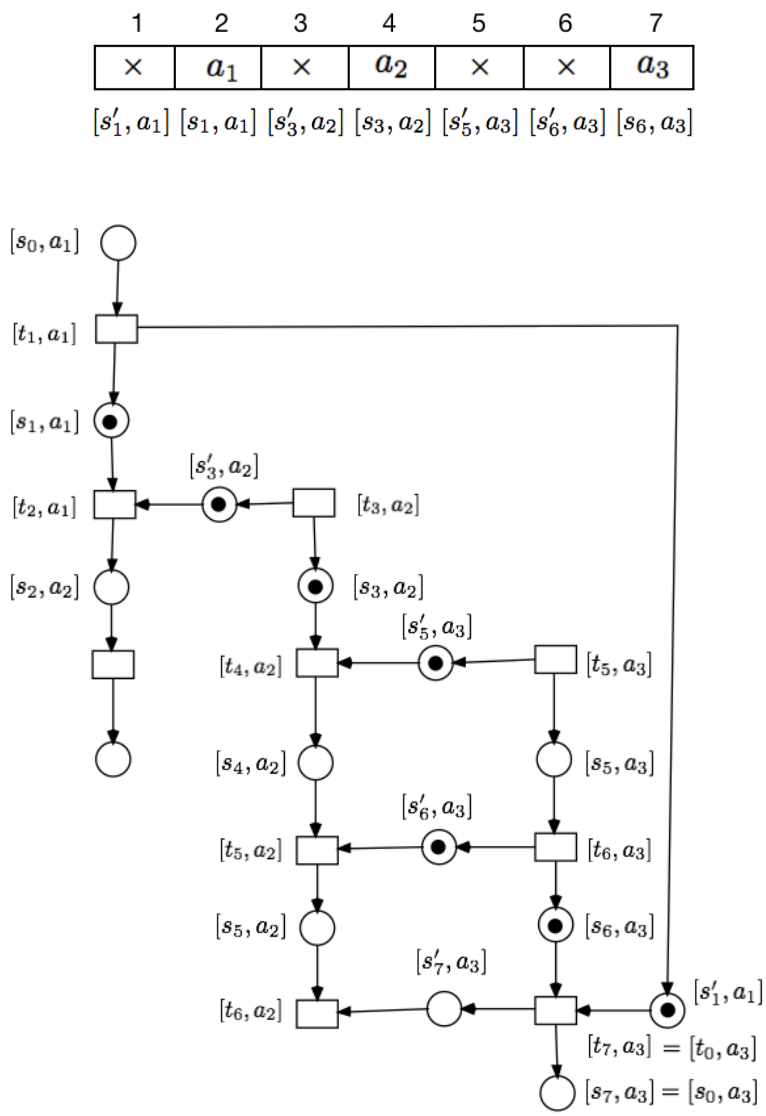

Fig. 13. Example for the bijections φ and ψ in the proof of part a) of Theorem 31.

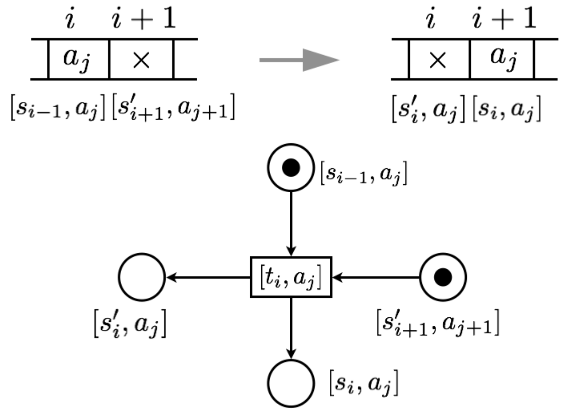

Fig. 14. See proof of Theorem 31, part a)

iii) $(\mathbf{s_1}, \mathbf{t}, \mathbf{s_2}) \notin \mathbf{tr} \Rightarrow (\varphi(\mathbf{s_1}), \psi(\mathbf{t}), \varphi(\mathbf{s_2})) \notin \mathbf{tr}'$.
A move of a traffic item a_j at position i is impossible if and only if at position $i+1$[5] is a traffic item a_{i+1}. This traffic item cannot have a move if there is a traffic item at position $(i+2)$ *mod* n. Any case, there is a sequence $a_j a_{j+1}, \cdots, a_w$ without gaps in between, but a gap at position $(w+1)$ *mod* n. By the map φ this corresponds to the following sequence of transitions and places in the cycloid:
$[t_i, a_j][s'_{i+1}, a_{j+1}], \cdots, [t_{v\ mod\ n}, a_{w\ mod\ c}][s'_{(v+1)\ mod\ n}, a_{(w+1)\ mod\ c}]$
In this sequence all but the last place are unmarked and the last place $[s'_{(v+1)\ mod\ n}, a_{(w+1)\ mod\ c}]$ is marked. The sequence is part of the release message chain of Figure 9. Since the cycloid is live and save (Theorem 21) the chain is part of a cycle with exactly one token, which implies that $[s'_{i+1}, a_{j+1}]$ is unmarked. Transition $[t_i, a_j]$ cannot occur since it has $[s'_{i+1}, a_{j+1}]$ as a input place.

iv) **Initial state.** By Definition 1 the regular initial state of tq-$1(c, g)$ is given by $ind(i) = a_i$ for $1 \le i \le c$ and $ind(i) = g_i$ for $c < i \le n$. Applying the mapping φ we obtain $ind'(i) = [s_{i-1}, a_i]$ for $1 \le i \le c$ and $ind'(i) = [s'_i, a_1]$ for $c < i \le n$. Hence, for $1 \le i \le c$ traffic item a_i is in position i and for $c < i \le n$ there is a gap and the next traffic item *modulo* n is a_1.
A standard initial state of a circular traffic queue with gaps has been defined in Definition 3 by the form $ind(1)ind(2)\cdots ind(n) = a_1{\times}^{r_1} a_2{\times}^{r_2} \cdots a_c{\times}^{r_c}$ with $a_i \in C, G = \{\times\}$ and $r_i = |\{\ x \in \mathbb{N} \mid i-1 < \frac{c}{g} \cdot x \le i\}|$ for $1 \le i \le c$. We have to prove that this meets Definition 15 of a standard initial marking of the cycloid. The parameter i in the definition of r_i takes the role of η-coordinate in the corresponding fundamental parallelogram of the cycloid, but taken in

[5] Recall the convention that all indices containing i are understood *modulo* p and all indices containing j are understood *modulo* c.

negative form. Nothing is changed in the definition of r_i, if we substitute the bound variable i by $-\eta$ and the bound $1 \leq i \leq c$ by $-1 \leq \eta \leq -c$. To reach Definition 15 we also substitute the parameters g, c and x by the general cycloid parameters α, β and ξ, respectively. As a result we obtain $-\eta - 1 < \frac{\beta}{\alpha} \cdot \xi \leq -\eta$ which is equivalent to the definition of the backward places $s^{\leftarrow}_{\xi,\eta} \in S^{\leftarrow}_1$ of M_0 in Definition 15: $\beta\xi + \alpha\eta \leq 0 \wedge \beta\xi + \alpha(\eta+1) > 0$. As in this definition the bound $-1 \leq \eta \leq -c$ is omitted and is replaced by the quotient $/_{\equiv}$. The standard initial state is completely determined by the distribution of the gaps by the definition of the r_i. The same holds for the cycloid. In fact, the definition of the forward places $s^{\rightarrow}_{\xi,\eta} \in S^{\rightarrow}_1$ is unambigously deducible from the definition of the backward places $s^{\leftarrow}_{\xi,\eta} \in S^{\leftarrow}_1$.

Part b) of the theorem:

Let us first recall some modulo-identities and notations:

a) $(a \otimes b) \mathit{mod}\, n = (a \mathit{mod}\, n \ \otimes b \mathit{mod}\, n) \mathit{mod}\, n$ for $\otimes \in \{+, -, \cdot\}$
b) $z \mathit{mod}\, p = z - p \cdot \lfloor \frac{z}{p} \rfloor$
c) If $0 < z \leq p$ then $-z \mathit{mod}\, p = -z - p \cdot \lfloor \frac{-z}{p} \rfloor = -z + p \cdot \lceil \frac{z}{p} \rceil = p - z$
d) $z \mathit{mod}\, p = r$ is also notated as $z \equiv r\ (\mathit{mod}\, p)$
e) $(z + k) \mathit{mod}\, p = z \mathit{mod}\, p$ for all $k \in \mathbb{N}, z \in \mathbb{Z}$

In this part we start with a labelled transition system $LTS_p(c, g)$ (Definition 6) with $r = \frac{g}{\Delta}$ and $p = r \cdot n$.

i) **Definition of φ and ψ.** In particular two enhancements are to be performed: the extension of the set of transitions and the introduction of traffic items $u_j \in G$. The first one is inherited from $LTS_p(c, g)$ by $\psi(\langle\!\langle t_i, a_j \rangle\!\rangle) = [t_v, a_j]$ with $v = k \cdot n + i$ when a_j in position i is moved. The latter is done by replacing in s'_i the (anonymous) apostrophe by u_j in the form $s_i^{u_j}$ Thus we obtain the form $s_i^{u_j}$, as shown in the example in top of Figure 15. It is important to note that two different traffic items $u_j, u_k \in G$ never share the same place in $\mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta}, M_0)$. This property holds since $\alpha = g$ is a divisor of $\gamma = \frac{g \cdot c}{\Delta}$ and is proved as the corresponding relation for regular cycloids where $\beta | \delta$. A naming by $s_i^{u_j}$ indicates a typing of this place by the traffic item u_j or by transformation to the dual cycloid. This property of cycloids can be called *backward-regular*. The length of such backward processes is $p' = \frac{A}{g} = \frac{c}{\Delta}(g + c)$. Therefore the traffic items from G need $\frac{c}{\Delta}$ rounds before reaching the initial state again. To trace the history of such traffic items we use an attribute x in the form u_h^x. Traffic item $u_h, 1 \leq h \leq g$ starts with $x = c + h$, as $c + h$ is its position in the regular initial marking and continues counting downwards $u_h^{c+h}, u_h^{c+h-1}, u_h^{c+h-2}, \cdots u_h^0, \cdots$ down to $u_h^{-p'+c+h}$ and then is reset to u_h^{c+h} and repeats. This done by the following modification of the transition rule in Definition 6:

$$
\begin{aligned}
&(ind_1, [t_v, a_j^k], ind_2) \in tr \Leftrightarrow \\
&\exists\, i \in \{1, \cdots, n\} : v = k \cdot n + i \ \wedge \\
&ind_1(i) = a_j^k \in C^r \ \wedge \ ind_1((i+1)\, mod\; n) = u_h^x \in G \wedge \\
&ind_2(i) = u_h^{x'} \text{with } \mathbf{if}\ x' = c + h - (p' - 1)\ \mathbf{then}\ c + h\ \mathbf{else}\ x - 1 \wedge \\
&ind_2((i+1)\, mod\; n) = a_j^k\ \mathbf{if}\ i \neq j\ \mathbf{else}\ a_j^{k+1} \wedge \\
&ind_2(l) = ind_1(l) \ \text{ for all } \ l \notin \{i, (i+1)\, mod\; n\}
\end{aligned}
\tag{5}
$$

and the regular initial marking to $ind(i) = a_i$ for $1 \leq i \leq c$ and $ind(i) = g_{i-c}^i$ for $c+1 \leq i \leq n$. Adding this attribute results in an isomorphic transition system since a number of $p' = \frac{\Xi(c,g)}{g} = \frac{c}{\Delta} \cdot (c+g)$ counter values is distributed on an at least as large path of length $\Xi(c,g) \geq p'$ in the transition system (Theorem 5). The mapping φ is defined by extending the corresponding definition in part a):
For $1 \leq i \leq n$ let be $ind'(i) := [\hat{s}_i, next(i)]$ with
$\hat{s}_i = \begin{cases} s_{(v-1)\, mod\, p} & \text{if } ind(i) = a_j^k \in C \\ s_{\rho_h(x)}^x & \text{if } ind(i) = u_h^x \in G \end{cases}$ where $v = k \cdot n + i$ and $next(i)$ is defined as before. The (partial) function $\rho_h(x)$ determines the place $[s_{\rho_h(x)}^{u_h}, a_j]$ (for some j) of traffic item $u_h \in G$ after x steps when starting in its initial position $[s_{\rho_h(c+h)}^{u_h}, a_j]$ in the standard initial marking.
In the case $h = 1$ the partial function $\rho_1(x)$ is defined on a number of p' values $c+1, c, c-1, c-2, \cdots, 1, 0, -1, \cdots - p' + c + 2$. The place $[s_{\rho_1(c+1)}^{u_1}, a_1]$, corresponding to the first of these arguments, is the backward input place of transition $[t_c, a_c]$. Hence by Corollary 26 this place is $[s_{p-\alpha+1}^{u_1}, a_1]$, where the apostrophe is replaced by u_1. Therefore $\rho_1(c+1) = p - \alpha + 1$ with $\alpha = g$. The next place of u_1 (determined by $\rho_1(c)$) is $[t_\beta, a_\beta]^{\leftarrow} = [s_\beta, a_\beta] = [s_c^{u_1}, a_c]$ and therefore $\rho_1(c) = c$.
Now let us recompute the difference between these two values $modulo\, p$. By identity a) we obtain: $\rho_1(c) - \rho_1(c+1) \equiv c\, mod\, p - (p - g + 1)\, mod\, p \equiv (c - (p - g + 1)) \equiv (c + g) - p - 1 \equiv n - p - 1 (\, mod\, p)$. As $p - n + 1 < p$ we obtain $n - p - 1\, mod\, p = -(p - n + 1)\, mod\, p = p - (p - n + 1) = n - 1$ (identity c). We call this (positive) difference in the value a **huge hop**.
The next $c - 1$ values are $\rho_1(c-1) = c - 1, \rho_1(c-2) = c - 2, \cdots \rho_1(1) = 1$. Each of these (negative) differences is called a **small hop**. The corresponding places are $[s_{c-1}^{u_1}, a_{c-1}], [s_{c-2}^{u_1}, a_{c-2}], \cdots, [s_1^{u_1}, a_1]$. We call this subsequence a *round*, which starts with a huge hop in the values of ρ followed by $c-1$ small hops. The total difference of a round is $\rho_1(1) - \rho_1(c+1) = (n-1) + (c-1) \cdot (-1) = n - 1 - c + 1 = g + c - c = g$. As there are $\frac{p'}{c}$ rounds the total difference is $(\frac{p'}{c} \cdot g)\, mod\, p = (\frac{A}{g} \cdot \frac{1}{c} \cdot g)\, mod\, p = (\frac{A}{c})\, mod\, p = p\, mod\, p = 0$ and the initial value is reached again.
Next we prove a compact formula for ρ_h for the position of traffic item $g_h \in G$, referring to the special case $\rho_1(x)$:

$\rho_h(x) = (\rho_1(x - (h-1)) + (h-1)$ for $1 \leq h \leq g)\, mod'\, p$ and
$\rho_1(x) = ((\frac{1}{c}(n \cdot (x\, mod'\, c) - g \cdot x)\, mod\, p$
for $c + 1 \geq x > -p' + c + 2$, $1 \leq h \leq g$.
The function $x\, mod'\, c$ is similar to $x\, mod\, c$, but returns c when $x\, mod\, c = 0$:
$x\, mod'\, c \; = \begin{cases} c & \text{if } x\, mod\, c = 0 \\ x\, mod\, c & \text{otherwise} \end{cases}$
We prove the formular for $h = 1$. Due to the periodic values of the function ρ_1 its arguments x are parametrized by k and i in the form $x = c+1-k\cdot c-i$ with $0 \leq k < \frac{p'}{c}$ and $0 \leq i < c$. Due to the modified function mod' there are two cases to distinguish for ρ_1:

a) case $i = 0$
$\rho_1(x) \equiv \rho_1(c + 1 - k \cdot c) \equiv$
$((\frac{1}{c}(n \cdot ((c + 1 - k \cdot c)\, mod'\, c) - g \cdot (c + 1 - k \cdot c)) \equiv$
$((\frac{1}{c}(n \cdot (1\, mod'\, c) - g \cdot (c + 1 - k \cdot c)) \equiv$
$((\frac{1}{c}((g + c) \cdot 1 - g \cdot c - g + g \cdot k \cdot c)) \equiv$
$((\frac{1}{c}(c - g \cdot c + g \cdot k \cdot c)) \equiv$
$1 - g + g \cdot k\, (mod\, p)$

b) $i \neq 0$
$\rho_1(x) \equiv \rho_1(c + 1 - k \cdot c - i) \equiv$
$((\frac{1}{c}(n \cdot ((c + 1 - k \cdot c - i)\, mod'\, c) - g \cdot (c + 1 - k \cdot c - i)) \equiv$
$((\frac{1}{c}(n \cdot ((1 - i)\, mod'\, c) - g \cdot (c + 1 - k \cdot c - i)) \equiv^{*)}$
$(\frac{1}{c}((g + c) \cdot (c - i + 1) - g \cdot c - g + g \cdot k \cdot c - i) \equiv$
$\frac{1}{c}(c \cdot c - c \cdot i + c + g \cdot k \cdot c) \equiv$
$c - i + 1 + g \cdot k\, (mod\, p)$
In case of the congruence $\equiv^{*)}$ there are two subcases
$b_1) : i \neq 1$ and $b_2) : i = 1$:
for $b_1)$ since $i < c$ we obtain $(1 - i)\, mod'\, c = (1 - i)\, mod\, c = -(i - 1)\, mod'\, c = c - i + 1$
for $b_2)$ since $i = 1$ we obtain $(1 - i)\, mod'\, c = 0\, mod'\, c = c = c - i + 1$

With these two results we prove that values of $\rho_1(x)$ are characterized by a first hop of $n - 1$ (after the initial value, as shown above) followed by $c - 1$ hops of -1. This sequence is repeated $\frac{p'}{c}$ times.
There are three cases to be distinguished:

I) Step from $i = 0$ to $i = 1$: A huge hop of $\mathbf{n-1}$ is to be proved:
$\rho_1(c+1-k\cdot c-1)-\rho_1(c+1-k\cdot c) = c = c-1+1+g\cdot k-(1-g+g\cdot k) = c - 1 + g = \mathbf{n-1}$.

II) Step from i to $i + 1$ for $i \in \{1, \cdots, c-2\}$: A small hop of $-\mathbf{1}$ is to be proved: $\rho_1(c + 1 - k \cdot c - (i + 1)) - \rho_1(c + 1 - k \cdot c - i) = c - (i + 1) + 1 + g \cdot k - (c - i + 1 + g \cdot k) = -\mathbf{1}$

III) Step from $i = c - 1$ to $i = 0$ while k is incresed by 1: A small hop of $-\mathbf{1}$ is to be proved: $\rho_1(c + 1 - k \cdot c - (i + 1)) - \rho_1(c + 1 - k - i) = 1-g+g\cdot(k+1)-(c-(c-1)+1+g\cdot k) = 1-g+g\cdot k+g-c+c-1-1+g\cdot k = -\mathbf{1}$.

In top of Figure 15 a state of $LTS_{28}(3, 4)$ and $\mathcal{C}(4, 3, 12, 12)$ are given. For the partial state $ind(2) = a_1^1$, hence $k = 1, i = 2$, by the map φ we obtain

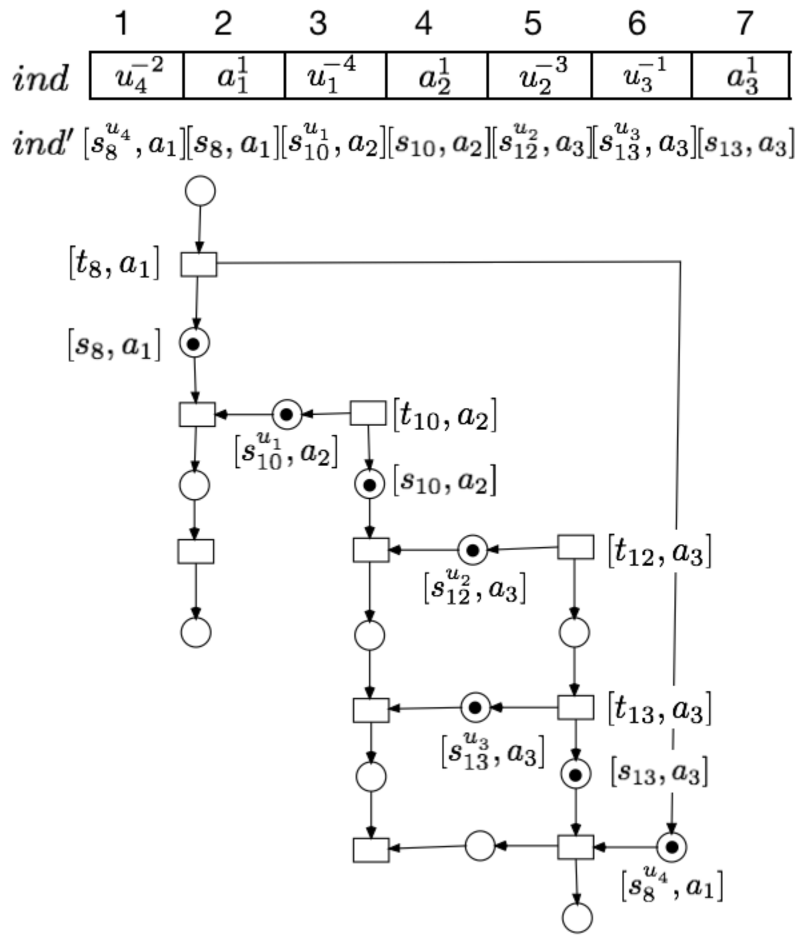

Fig. 15. Example for the bijections φ and ψ in the proof of part b) of Theorem 31.

$ind'(2) = [s_{(v-1)\,mod\,p}, a_1] = [s_8, a_1]$ since $v - 1 = k \cdot n + i - 1 = 1 \cdot 7 + 2 - 1 = 8$. Similarily, for $ind(3) = u_1^{-4}$ we obtain $x = -4$ in $\rho_1(-4) = (\frac{1}{3}(7 \cdot (-4\,mod'\,3) - 4 \cdot (-4)))\,mod\,28 = (\frac{1}{3}(7 \cdot 2 - 4 \cdot (-4))\,mod\,28 = 10$ which defines the place $[s_{10}^{u_1}, a_2]$. The cycloid is given in Figure 17 together with all values of ρ_h. In the net the path for u_4 corresponding to ρ_4 is marked by a particular colour of the net elements and bold arrows. Places like $[s_{24}^{u_4}, a_1]$ are represented as [s24-u4,a1].

ii) $(\mathbf{s_1}, \mathbf{t}, \mathbf{s_2}) \in \mathbf{tr} \Rightarrow (\varphi(\mathbf{s_1}), \psi(\mathbf{t}), \varphi(\mathbf{s_2})) \in \mathbf{tr}'$.
A transition from a state component $ind(i) = a_j^k$ implies that a traffic item a_j^k in position i is moving to position $i+1$[6]. This is possible if position $i+1$ is occupied by a traffic item $u_h^x \in G$ (see Figure 16). Hence two positions are involved: $ind(i)ind(i+1) = a_j^k u_h^x$ and this is changed by transition $[t_v, a_j^k]$ to $ind(i)ind(i+1) = u_h^{x'} a_j^k$ or $ind(i)ind(i+1) = u_h^{x'} a_j^{k+1}$ where x' is defined in equation 5.
With the mapping φ, as defined in i), we obtain that $ind'(i)ind'(i+1) = [s_{v-1}, a_j][s_{\rho(x)}^{u_h}, a_{j+1}]$ is changed to $ind'(i)ind'(i+1) = [s_{\rho(x')}^{u_h}, a_j][s_v, a_j]$. We have to prove that this is the modification performed by the transition $[t_v, a_j]$ in the regular cycloid, as by Definition 24 this transition has the output places $\{[s_v, a_j], [s_v^{u_h}, a_j]\}$, where the attribute u_h replaces the apostrophe and by Lemma 25 the input places $\{[s_{v-1}, a_j], [s_{v+1}^{u_h}, a_{j+1}]\}$.
Hence it remains to prove that the indices v of t_v and $\rho(x)$ are consistent. Formally in the case of u_1 (the other cases are similar) we have to prove that

$$[t_v, a_j]^{\leftarrow} = [s_{\rho(x)}^{u_1}, a_j] \;\Rightarrow\; \rho(x) = v \tag{6}$$

We first prove the following fact. If a transition $[t_v, a_j]$ exchanges a_j with u_h the next time in the same process $[t_{v'}, a_j]$ exchanges a_j with u_h is given by $v' = (v + g)\,mod\,p$. By Lemma 25 the index v is reduced a number of $\beta - 1$ times by 1 and once increased by $\alpha + \beta - 1$. In total we obtain : $v' = v + (\alpha + \beta - 1) - (\beta - 1) = v + \alpha = v + g$. By this intermediate fact for a fixed traffic item a_j it interchanges with u_1 by the transitions $[t_j, a_j], [t_{j+g}, a_j], [t_{j+2\cdot g}, a_j], \cdots, [t_{j+(\frac{p'}{c}\cdot g)\,mod\,p}, a_j] = [t_j, a_j]$. The implication (6) is proved by induction on this sequence:

Induction Basis: For $[t_j, a_j]$ we have $[t_j, a_j]^{\leftarrow} = [s_j^{u_1}, a_j]$ and therefore $\rho(x) = \rho(j) = j$. $x = j$ holds as the counter x is initiated with $x_0 = c + 1$ and is decremented $c - j + 1$ times, hence $x = c + 1$-(c-j+1) = j

Induction Hypothesis: $[t_{j+r\cdot g}, a_j]^{\leftarrow} = [s_{\rho(x)}^{u_1}, a_j] \;\Rightarrow\; \rho(x) = j + r \cdot g$

Induction Step: $[t_{j+(r+1)\cdot g}, a_j]^{\leftarrow} = [s_{\rho(x)}^{u_1}, a_j] \;\Rightarrow\; \rho(x) = j + (r+1) \cdot g$ It has been proved before that by a round the value of $\rho(x)$ is increased by g. Therefore $\rho(x) = j + r \cdot g + g = j + (r+1) \cdot g$

iii) $(\mathbf{s_1}, \mathbf{t}, \mathbf{s_2}) \notin \mathbf{tr} \Rightarrow (\varphi(\mathbf{s_1}), \psi(\mathbf{t}), \varphi(\mathbf{s_2})) \notin \mathbf{tr}'$.
In this part all indices containing i, j, h, v are understood $mod\,p$, $mod\,c$, $mod\,g$, $mod\,p$ respectively. A move of a traffic item a_j at position i is impossible if and only if at position $i+1$ is a traffic item a_{i+1}. This traffic item

[6] $mod\,n$ is omitted here and in the following.

cannot have a move if there is a traffic item at position $i+2$. Any case, there is a sequence $a_j a_{j+1}, \cdots, a_w$ without gaps in between, but a gap at position $(w+1)$ *mod* n. By the map φ this corresponds to the following sequence of transitions and places in the cycloid:
$[t_v, a_j][s^{u_h}_{v+1}, a_{j+1}], \cdots, [t_{v+r}, a_{j+r}][s^{u_{h+r}}_{(v+r+1)\ mod\ n}, a_{j+r+1}]$
In this sequence all but the last place are unmarked and the last place is marked. The sequence is part of the release message chain of Figure 9. Since the cycloid is live and save (Theorem 21) the chain is part of a cycle with exactly one token, which implies that $[s^{u_h}_{v+1}, a_{j+1}]$ is unmarked. Transition $[t_v, a_j]$ cannot occur since it has $[s^{u_h}_{v+1}, a_{j+1}]$ as a input place.

iv) **Initial state.**

The proofs respective to initial markings are the same as in part a) as the initial partes of the cycloids are the same. □

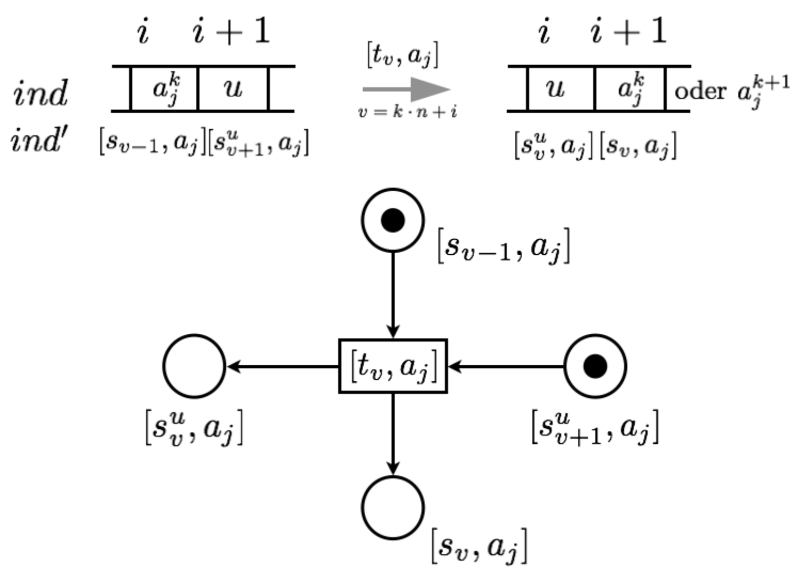

Fig. 16. See proof of Theorem 31, part b)

In Figure 18 by $\mathcal{C}(4, 6, 12, 12)$ an example of a cycloid is given with $\Delta = 2$. Its parameters are $A = 120, cyc = p = 20, rm = 22$.

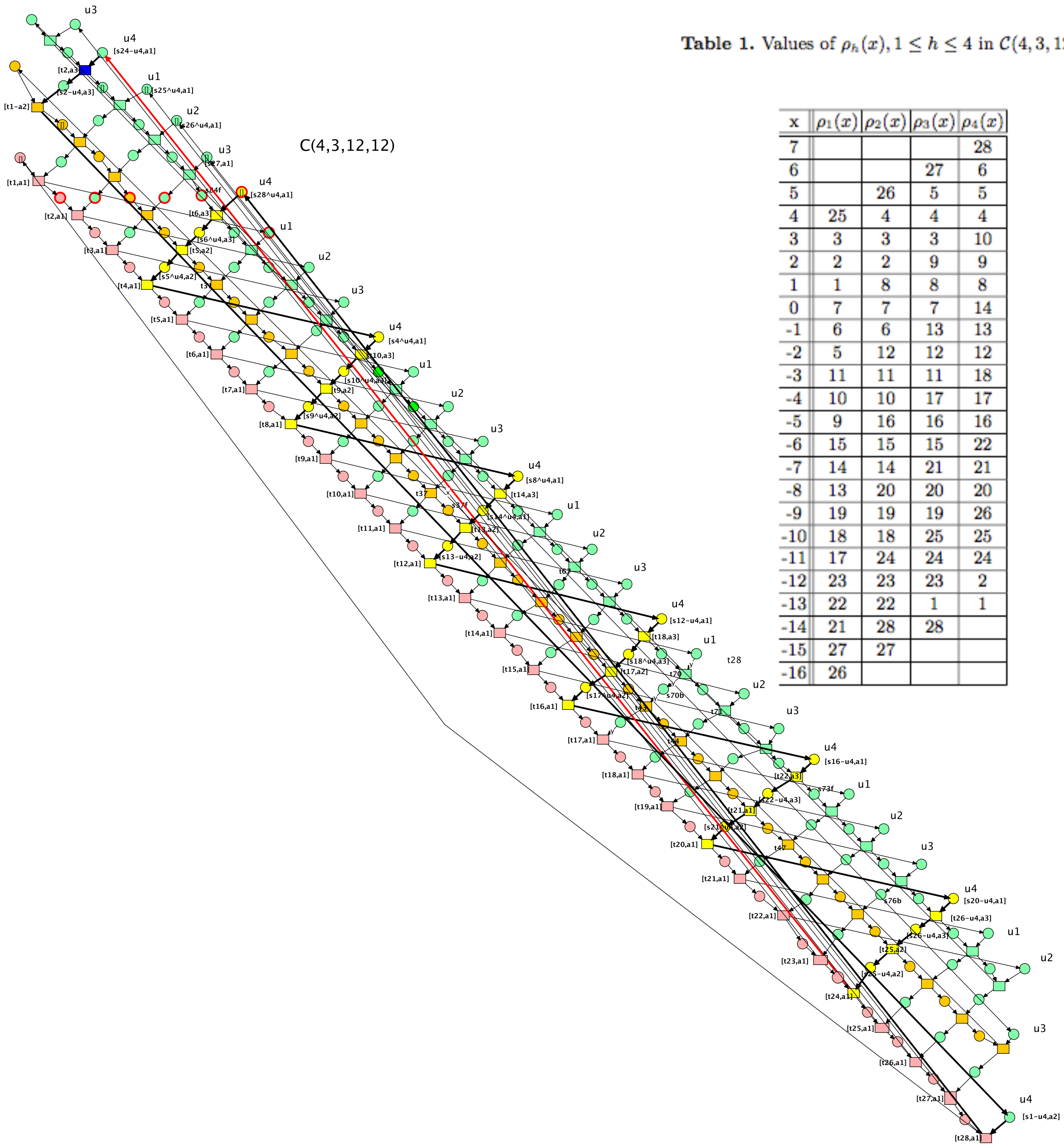

Table 1. Values of $\rho_h(x), 1 \leq h \leq 4$ in $\mathcal{C}(4,3,12,12)$

x	$\rho_1(x)$	$\rho_2(x)$	$\rho_3(x)$	$\rho_4(x)$
7				28
6			27	6
5		26	5	5
4	25	4	4	4
3	3	3	3	10
2	2	2	9	9
1	1	8	8	8
0	7	7	7	14
-1	6	6	13	13
-2	5	12	12	12
-3	11	11	11	18
-4	10	10	17	17
-5	9	16	16	16
-6	15	15	15	22
-7	14	14	21	21
-8	13	20	20	20
-9	19	19	19	26
-10	18	18	25	25
-11	17	24	24	24
-12	23	23	23	2
-13	22	22	1	1
-14	21	28	28	
-15	27	27		
-16	26			

Fig. 17. Cycloid $\mathcal{C}(4,3,12,12)$ with regular coordinates.

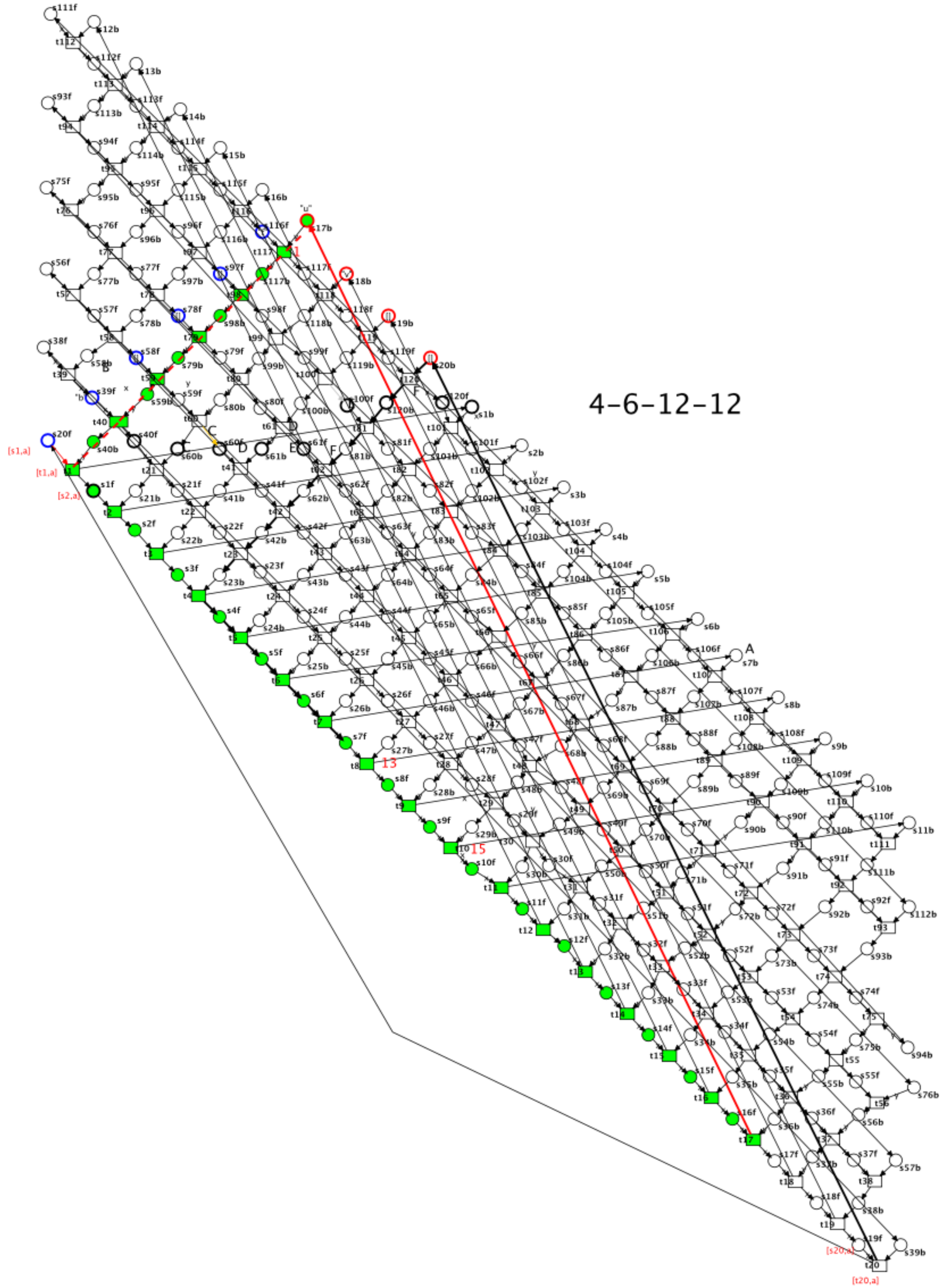

Fig. 18. Cycloid $\mathcal{C}(4,6,12,12)$ with $\Delta = 2$.

4.4 Representations of circular queues by T-Nets and Coloured Nets

In Figure 19a) Petri's queue of cars is represented as a net with a regular initial marking. The positions of a number of c cars are represented by black tokens in the places $s_0, \cdots, s_{c-1}$, followed by g tokens in the complementary places $s'_{c+1}, \cdots, s'_{c+g}$ representing the gaps. By the complementary places the net is safe and the cars cannot pass each other.

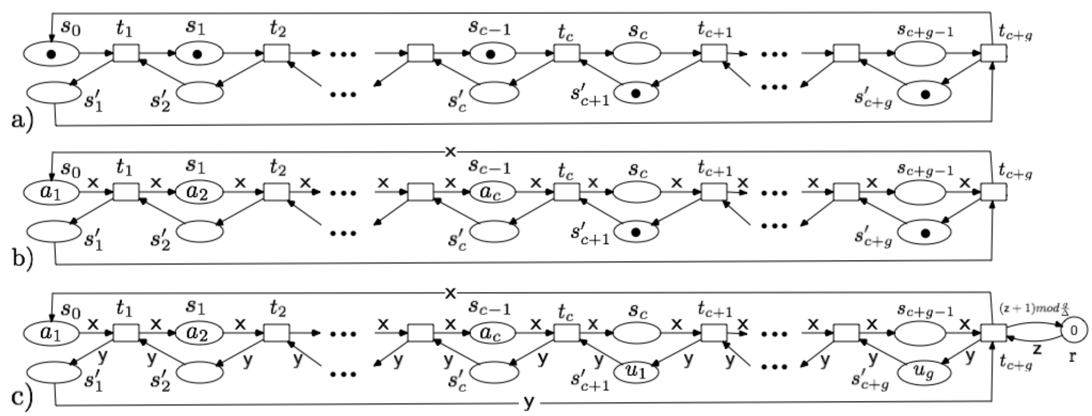

Fig. 19. Nets of circular traffic queues a) $\mathcal{N}_{basic}(c, g)$ b) $\mathcal{N}_{coul}(c, g)$ and c) $\mathcal{N}_{sym}(c, g)$.

They cannot be distinguished by their identifiers, which is different in part b) of Figure 19, where the cars have identifiers $a_1, \cdots, a_c$. To handle such individual tokens a coloured net is used containing a variable x. As shown, the net has the behaviour of a *circular traffic queue with gaps tq*-1(c, g).

In the next step in Figure 19c), a *circular traffic queue tq*-2(c, g) is modelled by replacing the undistinguishable gap tokens in $s'_{c+1}, \cdots, s'_{c+g}$ by identifiers $u_1, \cdots, u_g$.

Definition 32. *a) A* basic tq-net $\mathcal{N}_{basic}(c, g) = (S, T, F, M_0)$ *is defined as follows by using the abbreviation* $p := c + g$ *(see Figure 19 a):*
$S := \{s_0, \cdots, s_{p-1}\} \cup \{s'_1, \cdots, s'_p\}$, $T := \{t_1, \cdots, t_p\}$ *and* $F := F^1 \cup F^2 \cup F^3 \cup F^4$ *where* $F^1 := \{(s_i, t_{(i+1) mod\, p}) | 0 \le i < p\}$, $F^2 := \{(t_i, s_{i\, mod\, p})) | 1 \le i \le p\}$, $F^3 := \{(s'_i, t_{(i-1) mod\, p})) | 1 \le i \le p\}$, $F^4 := \{(t_i, s'_i) | 1 \le i \le p\}$,
$M_0 := \{s_0, \cdots, s_{c-1}\} \cup \{s'_{c+1}, \cdots, s'_p\}$

b) In a coloured tq-net $\mathcal{N}_{coul}(c, g) = (S, T, F, var, M_0)$ *the sets* S, T *and* F *are defined as in part a), but* $var(F_1 \cup F_2) := x$, $M_0(s_i) = a_{i+1}$ *for* $0 \le i < c$ *and* $M_0(s'_i) = \bullet$ *for* $c < i \le p$ *(see Figure 19 b).*

c) *In a* symmetric coloured tq-net $\mathcal{N}_{sym}(c,g) = (S,T,F,var,M_0)$ *the sets* S,T *and* F *are defined as in part a), but* $var(F_1 \cup F_2) := x$, $var(F_3 \cup F_4) := y$, $M_0(s_i) = a_{i+1}$ *for* $0 \leq i < c$ *and* $M_0(s'_i) = v_i$ *for* $c+1 \leq i \leq p$ *(see Figure 19 b).*

As we are interested in modelling traffic queues by T-nets we give a behaviour preserving transformation of such coloured tq-nets into T-nets. An almost standard construction of such a transformation into nets works as follows: for each place s of $\mathcal{N}_{coul}$ and coloured token a a new place $[s,a]$ is created, simulating the token a to be located in the place s of $\mathcal{N}_{coul}$. In a similar way, for each transition t of $\mathcal{N}_{coul}$ and coloured token a a new transition $[t,A]$ is created, simulating the token a to be moved by the transition t of $\mathcal{N}_{coul}$. Places with a black token as colour set are treated accordingly. This leads to the following Definition (see Figure 20 a)).

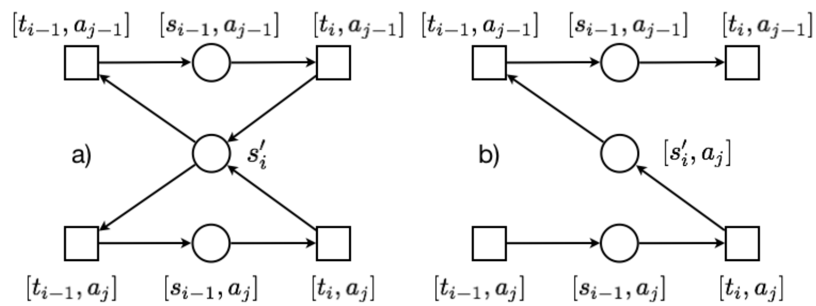

Fig. 20. Transformation of a coloured tq-net

Definition 33. *For a coloured tq-net* $\mathcal{N}_{coul}(c,g) = (S,T,F,var,M_0)$ *a net* $\mathcal{N}_0(c,g) = (S_0,T_0,F_0,M_0^0)$ *is defined as follows:* $p := c+g$
$S_0 := (\{s_0, \cdots, s_{p-1}\} \times C) \cup \{s'_1, \cdots, s'_p\}$, *where* $C = \{a_1, \cdots, a_c\}$ *is set of cars,*
$T_0 := T \times C$,
$F_0 := F_0^1 \cup F_0^2 \cup F_0^3 \cup F_0^4$
$F_0^1 := \{([s_i,a_j],[t_{(i+1) mod\, p},a_j]) | 0 \leq i < p, 1 \leq j \leq c\}$,
$F_0^2 := \{([t_i,a_j],[s_{i mod\, p},a_j]) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$F_0^3 := \{(s'_i,[t_{(i-1) mod\, p},a_j]) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$F_0^4 := \{([t_i,a_j],s'_i) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$M_0 := \{[s_i,a_i] | 0 \leq i < c\} \cup \{s'_i | 0 < i \leq p\}$

The net from Definition 33 is not a T-net (see Figure 20 a)). It can be easily transformed by observing the specification of a circular traffic queue of Definition 7 b): Each traffic item $a \in C$ can make a step from position $i \in \{1, \cdots, n\}$ to position $(i+1)\ mod\ n$, if it has received a permit signal from a gap in position

$(i+1)\ mod\ n$. After this step the traffic item is in position i. Hence, by replacing the place s'_i in Figure 20 a) by copies $[s'_i, a_j]$ $(1 \leq j \leq c)$, but omitting the arcs $([t_i, a_{j-1}], s'_i)$ and $(s'_i, [t_{i-1}, a_j])$ as shown in Figure 20 b) we obtain a T-net, with the same behaviour. It is formally given in Definition 33 and is identical to the regular net of Definition 24 and Lemma 25 with process length $p = g + c$.

Definition 34. *For a coloured tq-net* $\mathcal{N}_{coul}(c,g) = (S,T,F,var,M_0)$ *a net* $\mathcal{N}_1(c,g) = (S_1,T_1,F_1,M_1^0)$, *called T-net-equivalent of* $\mathcal{N}_{coul}(c,g)$, *is defined as follows:*
$p = c + g$, $S_1 := S \times C$ *and* $T_1 := T \times C$ *where* $C = \{a_1, \cdots, a_c\}$ *is set of cars,*
$F_1 := F_1^1 \cup F_1^2 \cup F_1^3 \cup F_1^4$
$F_1^1 := \{([s_i, a_j], [t_{(i+1) mod\, p}, a_j]) | 0 \leq i < p, 1 \leq j \leq c\}$,
$F_1^2 := \{([t_i, a_j], [s_{i\, mod\, p}, a_j]) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$F_1^3 := \{[(s'_i, a_j], [t_{(i-1) mod\, p}, a_{j-1\, mod\, c}]) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$F_1^4 := \{([t_i, a_j], [s'_i, a_j]) | 1 \leq i \leq p, 1 \leq j \leq c\}$,
$M_0^1 := \{[s_i, a_{(i+1)\, mod\, c}] | 0 \leq i < c\} \cup \{[s'_i, a_1] | p - g + 1 \leq i \leq p\}$

Theorem 35. *The T-net-equivalent of* $\mathcal{N}_{coul}(c,g)$ *from Definition 34 is isomorphic (with identical names) to the regular cycloid of Definition 24 and Lemma 25 with process length* $p = g + c$. *By Theorem 29 it is also isomorphic to the cycloid* $\mathcal{C}(g,c,c,c)$.

Proof. The identical forms of the net from Definition 34 and of the regular cycloid of Definition 24 and Lemma 25 with process length $p = g + c$ can be immediately checked (also compare with Figure 14). By the parameters of this cycloid in Theorem 29 it is proved to be isomorphic to $\mathcal{C}(g,c,c,c)$. As the places $s_0, s_1, \cdots, s_{c-1}$ are marked by $a_1, a_2, \cdots, a_c$ in $\mathcal{N}_{coul}(c,g)$ the same holds for $[s_0, a_1], [s_1, a_2], \cdots, [s_{c-1}, a_c]$ in $\mathcal{N}_1(c,g)$. In the same way, as the places $s'_{c+1}, s'_{c+2}, \cdots, s'_{c+g}$ are marked in $\mathcal{N}_{coul}(c,g)$ the same holds for $[s'_{p-g+1}, a_1], [s'_{p-g+2}, a_1], \cdots, [s'_p, a_1]$ in $\mathcal{N}_1(c,g)$. Note that $p - g + 1 = g + c - g + 1 = c + 1$. As before, for black tokens, we have a different construction rule: a_1 is the second component, since the token represents the release message of a_1. □

The construction of the T-net-equivalent of $\mathcal{N}_{sym}(c,g)$ is similar to the preceding one. It is equivalent to the $\frac{g}{\Delta}$-fold iteration of $\mathcal{C}(g,c,c,c)$ (see Section 5). Within each copy of $\mathcal{C}(g,c,c,c)$ the construction is identical to Definition 35. At the borders the corresponding equivalent place is chosen, instead.

Definition 36. *For a coloured tq-net* $\mathcal{N}_{sym}(c,g) = (S,T,F,var,M_0)$ *a net* $\mathcal{N}_2(c,g) = (S_2,T_2,F_2,M_2^0)$, *called T-net-equivalent of* $\mathcal{N}_{sym}(c,g)$, *is defined as follows:*
$S_2 := S \times C \times D$ *and* $T_2 := T \times C \times D$ *where* $C = \{a_1, \cdots, a_c\}$ *is set of cars,*
$D = \{k \in \mathbb{N} | 0 \leq k < \frac{g}{\Delta}\}$ *and* $\tilde{p} = c + g$,
$F_2 := F_2^1 \cup F_2^2 \cup F_2^3 \cup F_2^4$
$F_2^1 := \{([s_i, a_j, k], [t_{i+1}, a_j, k]) | 0 \leq i < \tilde{p}, 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$,
$F_2^2 := \{([t_i, a_j, k], [s_i, a_j, k]) | 1 \leq i < \tilde{p}, 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$
$\cup \{([t_{\tilde{p}}, a_j, k], [s_0, a_j, (k+1) mod \frac{g}{\Delta}] | 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$

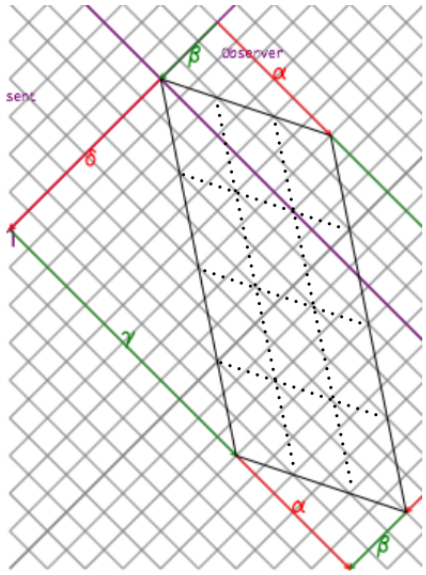

Fig. 21. $\mathcal{C}(6,3,12,8)$ with $a \cdot b = 3 \cdot 4$ copies of $\mathcal{C}(2,1,3,2)$ as substructures.

$F_2^3 := \{[(s'_i, a_j, k], [t_{(i-1) mod\, \tilde{p}}, a_{j-1\, mod\, c}, k])| 1 \leq i \leq \tilde{p}, 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$,
$F_2^4 := \{([t_i, a_j, k], [s'_i, a_j], k)| 1 \leq i \leq \tilde{p}, 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$,
$\cup\{([t_{\tilde{p}}, a_j, k], [s_{\tilde{p}}, a_j, (k+1) mod \frac{g}{\Delta}]| 1 \leq j \leq c, 0 \leq k < \frac{g}{\Delta}\}$
$M_0^2 := \{[s_i, a_{(i+1)\, mod\, c}]| 0 \leq i < c\} \cup \{[s'_i, a_1]| p - \alpha + 1 \leq i \leq p\}$

Theorem 37. *The T-net-equivalent of $\mathcal{N}_{sym}(c, g)$ from Definition 36 is isomorphic to the regular cycloid of Definition 24 and Lemma 25 with process length $p = \frac{g}{\Delta}(g+c)$. By Theorem 30 it is also isomorphic to the cycloid $\mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta})$.*

Proof. The identical forms of the net from Definition 36 and of the regular cycloid of Definition 24 and Lemma 25 with process length $p = \frac{g}{\Delta}(g+c)$ can be immediately checked (also compare with Figure 14). For the regular initial marking the current value of p has to be inserted. By the parameters of this cycloid in Theorem 30 it is proved to be isomorphic to $\mathcal{C}(g, c, \frac{g \cdot c}{\Delta}, \frac{g \cdot c}{\Delta})$. $\square$

5 Composition of Cycloids

Using Theorem 12 we next deduce that the equivalence relation $\equiv$ becomes finer if the cycloid parameters are integer multiples.

Theorem 38. *Let $\mathcal{C}_1(\alpha, \beta, \gamma, \delta)$ be a cycloid and $a, b \in \mathbb{N}_+$ such that a is a divisor of α and β as well as b is a divisor of γ and δ. Then the equivalence relation*

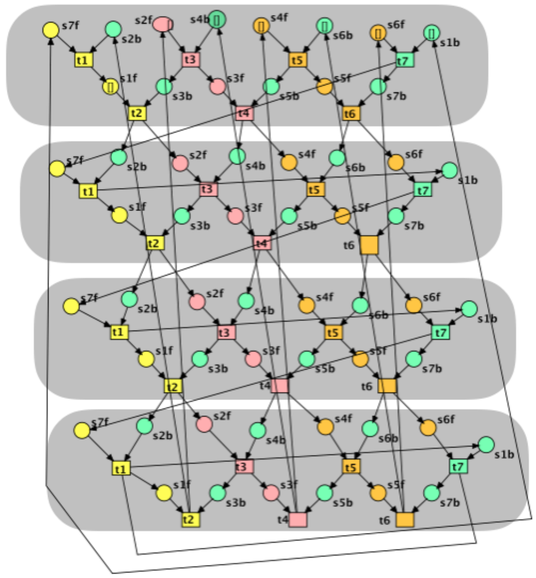

Fig. 22. Iteration $\mathcal{C}(3,4,1,1)^{[4]} \equiv \mathcal{C}(3,4,4,4)$

$\equiv_1$ of $\mathcal{C}_1(\alpha,\beta,\gamma,\delta)$ is included in the equivalence relation $\equiv_2$ of $\mathcal{C}_2(\frac{\alpha}{a},\frac{\beta}{a},\frac{\gamma}{b},\frac{\delta}{b})$, more precisely $\equiv_1\ \subseteq\ \equiv_2$.

Proof. Let be $\mathbf{v} = \mathbf{x_2} - \mathbf{x_1}$ and π_1 and π_2 the parameter vector function of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then we have to prove that $\pi_2(\mathbf{v}) \in \mathbb{Z}^2$ if $\pi_1(\mathbf{v}) \in \mathbb{Z}^2$. This is done y the following deduction, where A_1 is the area of $\mathcal{C}_1$ and $A_2 = \frac{1}{a \cdot b} A_1$ is the area of $\mathcal{C}_2$.

$$\pi_2(\mathbf{v}) = \frac{1}{A_2}\begin{pmatrix} \frac{\delta}{b} & \frac{-\gamma}{b} \\ \frac{\beta}{a} & \frac{\alpha}{a} \end{pmatrix}\mathbf{v} = \frac{a\cdot b}{A_1}\begin{pmatrix} \frac{\delta}{b} & \frac{-\gamma}{b} \\ \frac{\beta}{a} & \frac{\alpha}{a} \end{pmatrix}\mathbf{v} = \frac{1}{A_1}\begin{pmatrix} a\cdot\delta & -a\cdot\gamma \\ b\cdot\beta & b\cdot\alpha \end{pmatrix}\mathbf{v} =$$
$$\frac{1}{A_1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}\mathbf{v} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\frac{1}{A_1}\begin{pmatrix} \delta & -\gamma \\ \beta & \alpha \end{pmatrix}\mathbf{v} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\pi_1(\mathbf{v}).$$

Since $\pi_1(\mathbf{v}) \in \mathbb{Z}^2$ by assumption also $\pi_2(\mathbf{v}) \in \mathbb{Z}^2$.

Theorem 38 allows to define iterations of cycloids, both with respect to time and space, as shown in an example in Figure 21.

Definition 39. *For a cycloid $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ and positive integers $n,m \in \mathbb{N}_+$ the spaciotemporal iteration is defined by $\mathcal{C}(\alpha,\beta,\gamma,\delta)^{[n]}_{[m]} := \mathcal{C}(m\cdot\alpha, m\cdot\beta, n\cdot\gamma, n\cdot\delta)$. In particular, $\mathcal{C}(\alpha,\beta,\gamma,\delta)^{[n]} := \mathcal{C}(\alpha,\beta,\gamma,\delta)^{[n]}_{[1]}$ is the temporal iteration and $\mathcal{C}(\alpha,\beta,\gamma,\delta)_{[m]} := \mathcal{C}(\alpha,\beta,\gamma,\delta)^{[1]}_{[m]}$ is the spacial iteration.*

This allows to characterize the two most importand cycloids of this article by iterations of the cycloid $\mathcal{C}(\alpha,\beta,1,1)$.

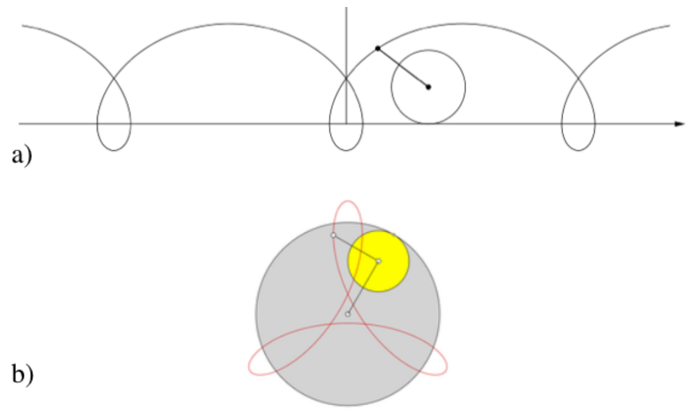

Fig. 23. Cycloid and epicycloid in geometry.

Lemma 40. *The iteration* $\mathcal{C}^{[n]}(\alpha,\beta,\gamma,\delta)$ *of a regular cycloid* $\mathcal{C}(\alpha,\beta,\gamma,\delta)$ *is regular with process length* $p^{[n]} = n \cdot p$ *if* p *is the process length of* $\mathcal{C}(\alpha,\beta,\gamma,\delta)$*.*

Proof. If β is a divisor of δ thean also of $n \cdot \delta$. If $A^{[n]}$ and A are the areas of the two cycloids then $A^{[n]} = \alpha \cdot n \cdot \delta + \beta \cdot n \cdot \delta = n \cdot A$ and $p^{[n]} = \frac{A^{[n]}}{\beta} = \frac{n \cdot A}{\beta} = n \cdot p$ □

Corollary 41. *a) The cycloid* $\mathcal{C}_1(g,c) = \mathcal{C}(g,c,c,c)$*, which is behaviour equivalent to the circular traffic queue tq-1*(c,g)*, is isomorphic to the c-fold temporal iteration of the basic cycloid:* $\mathcal{C}_0(g,c)^{[c]} = \mathcal{C}(g,c,1,1)^{[c]} \simeq \mathcal{C}(g,c,c,c)$*.*

b) The cycloid $\mathcal{C}_2(g,c) = \mathcal{C}(g,c,\frac{g \cdot c}{\Delta},\frac{g \cdot c}{\Delta})$*, which is behaviour equivalent to the circular traffic queue tq-2*(c,g)*, is isomorphic to the* $\frac{g \cdot c}{\Delta}$*-fold temporal iteration of the basic cycloid* $\mathcal{C}_0(g,c)^{[\frac{g \cdot c}{\Delta}]} = \mathcal{C}(g,c,1,1)^{[\frac{g \cdot c}{\Delta}]} \simeq \mathcal{C}(g,c,\frac{g \cdot c}{\Delta},\frac{g \cdot c}{\Delta})$*.*

c) The cycloid $\mathcal{C}_2(g,c)$ *is also isomorphic to the* $\frac{g}{\Delta}$*-fold temporal iteration of the cycloid* $\mathcal{C}_1(g,c)^{[\frac{g}{\Delta}]} \simeq \mathcal{C}_2(g,c)$*.*

Figure 22 shows the iteration $\mathcal{C}(3,4,1,1)^{[4]}$, which is isomorphic to $\mathcal{C}(3,4,4,4)$.

In geometry a cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping ([1], Figure 23 a)). An epicycloid or hypercycloid is a cycloid in circular form. It is a plane curve produced by tracing the path of a chosen point on the circumference of a circle, called an epicycle, which rolls without slipping around a fixed circle ([2], Figure 23 b)).

In [8] Petri introduces track and clock image of a cycloid: "To obtain a perceptual image of the Cycloid, we can paint the parallelogram on a rubber sheet and form it into a torus, pasting together first the top and bottom sides, and then the right and left. We call this a track image of the Cycloid: the track is a line which runs around the torus taking the long distance. If we paste together

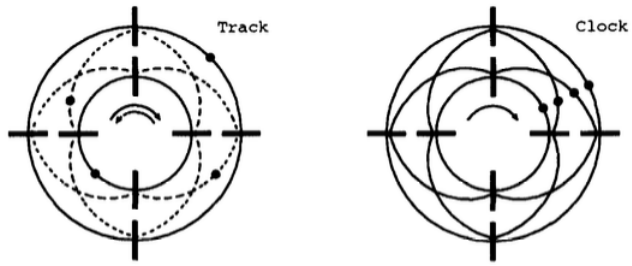

Fig. 24. Track and clock image of the cycloid $\mathcal{C}(2,2,2,2)$.

the right and left sides first, we obtain of course the same net topology, but a different image in $\mathbb{R}3$, called a clock image of the cycloid." Track and clock image of the cycloid $\mathcal{C}(2,2,2,2)$ from [8] are shown in Figure 24.

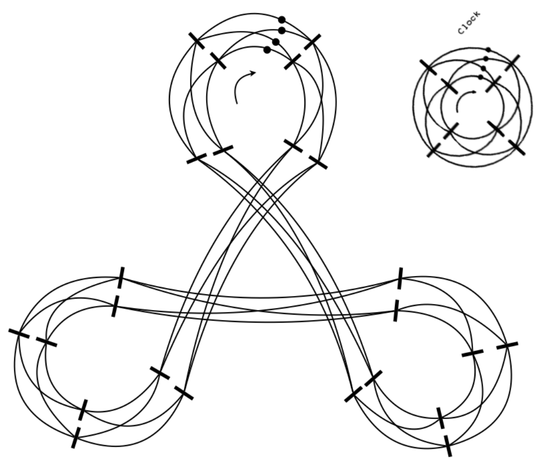

Fig. 25. $\mathcal{C}(2,2,2,2)^{[3]} \simeq \mathcal{C}(2,2,6,6)$
represented as epicycloid.

Similar to epicycloids iterated cycloid nets are twofold repetitive structures, namely repetitive in the component parts as well as in the overall structure. Hence we obtain an analogy as shown in Figure 25 as a 3-fold iteration $\mathcal{C}(2,2,2,2)^{[3]} \simeq \mathcal{C}(2,2,6,6)$ of the clock image of the cycloid $\mathcal{C}(2,2,2,2)$ from Figure 24.

6 Summary

The theory of cycloids has been extended by new formal methods and new results concerning circular traffic queues. The use of matrix algebra has lead to a more mathematical and easier handling of the cycloid equivalence relation. Regular cycloids have been shown to be a useful link in the theory. They are structural near to circular traffic queues, but miss some of the clear mathematical properties of general cycloids. The proof of isomorphism of circular traffic queues and special cycloids was facilitated by the use of regular cycloids as a link. The concept of release message chain and cycle has been introduced. It was found to be closely connected to the notion of minimal cycle which was so important in earlier publications on cycloids.

The most important results are summarized in Table 1: the two models of circular traffic queues $tq\text{-}1(c,g)$ and $tq\text{-}2(c,g)$, their modelling by cycloids $\mathcal{C}_0(g,c)$, $\mathcal{C}_1(g,c)$ and $\mathcal{C}_2(g,c)$, the corresponding values of minimal cycles and numbers of transitions and their representations as iterations of $\mathcal{C}_0(g,c)$.

Table 1. Summary of some results.

model	cycloid	minmal cycle cyc	no of transitions A	iterated cycloid
$tq\text{-}1(c,g)$	$\mathcal{C}_0(g,c) = \mathcal{C}(g,c,1,1)$	2	$g+c$	$\mathcal{C}_0(g,c)^{[1]}$
$tq\text{-}1(c,g)$	$\mathcal{C}_1(g,c) = \mathcal{C}(g,c,c,c)$	$\begin{cases} 2\cdot c & \text{if } g \geq c \\ g+c & \text{if } c > g \end{cases}$	$\Gamma(c,g) = (g+c)c$	$\mathcal{C}_0(g,c)^{[c]}$
$tq\text{-}2(c,g)$	$\mathcal{C}_2(g,c) = \mathcal{C}(g,c,\frac{g\cdot c}{\Delta},\frac{g\cdot c}{\Delta})$	$\begin{cases} \frac{c}{\Delta}(g+c) & \text{if } g \geq c \\ \frac{g}{\Delta}(g+c) & \text{if } c > g \end{cases}$	$\Xi(c,g) = \frac{g}{\Delta}(g+c)c$	$\mathcal{C}_0(g,c)^{[\frac{g\cdot c}{\Delta}]}$

References

1. https://en.wikipedia.org/wiki/Cycloid
2. https://en.wikipedia.org/wiki/Epicycloid
3. Desel, J., Esparza, J.: Free Choice Petri Nets. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press (1995)
4. Girault, C., Valk, R. (eds.): Petri Nets for System Engineering - A Guide to Modelling, Verification and Applications, 585 pages. Springer, Berlin (2003)
5. Kummer, O., Stehr, M.O.: Petri's Axioms of Concurrency - a Selection of Recent Results. In: Application and Theory of Petri Nets 1997. Lecture Notes in Computer Science, vol. 1248, pp. 195 – 214. Springer-Verlag, Berlin (1997)
6. Lang, S.: Algebra. Springer (1970)
7. Petri, C.A.: On Technical Safety and Security. Petri Net Newsletter 33, 25–30 (1989)

8. Petri, C.A.: Nets, Time and Space. Theoretical Computer Science (153), 3–48 (1996)
9. Petri, C.A., Valk, R.: On the Physical Basics of Information Flow - Results obtained in cooperation Konrad Zuse. `http://uhh.de/inf-petri-xian08` (2008)
10. Petri, C.A., Yuan, C.Y.: On Technical Safety and Security (continued). Petri Net Newsletter 35, 8–15 (1990)
11. Smith, E., Reisig, W.: The semantics of a net is a net – an exercise in general net theory. In: Voss, K., Genrich, J., Rozenberg, G. (eds.) Concurrency and Nets. pp. 461–479. Springer-Verlag, Berlin (1987)
12. Stehr, M.O.: Characterizing Security in Synchronization Graphs. Petri Net Newsletter 56, 17–26 (1999)
13. Valk, R.: On the Structure of Cycloids Indroduced by Carl Adam Petri. In: Application and Theory of Petri Nets and Concurrency. Lecture Notes in Computer Science, vol. 10877, pp. 294 – 314. Springer-Verlag, Berlin (2018)
14. Valk, R.: Formal Properties of Petri's Cycloid Systems. Fundamenta Informaticae 169, 85–121 (2019)